Young Mathematician's Guide:

Being a PLAIN and EASY

INTRODUCTION

TO THE

MATHEMATICKS.

In FIVE PARTS.

VIZ.

I. ARITHMETICK, Vulgar and Decimal, with all the useful Rules; and a general Method of extracting the Roots of all single Powers.

II. ALGEBRA, or Arithmetick in Species; wherein the Method of raising and resolving Equations is rendered easy; and illustrated with Variety of Examples, and numerical Questions. Also the whole Business of Interest and Annuities, &c. performed by the Pen.

III. The ELEMENTS of GEOMETRY contracted, and analytically demonstrated; with a new and easy Method of finding the Circle's Periphery and Area to any assigned Exactness, by one Equation only: Also a new Way of making Signs and Tangents.

IV. CONICK SECTIONS, wherein the chief Properties, &c. of the Ellipsis, Parabola, and Hyperbola, are clearly demonstrated.

V. The ARITHMETICK of INFINITES explained and rendered Easy; with its Application to superficial and solid Geometry.

With an APPENDIX of PRACTICAL GAUGING.

By *JOHN WARD.*

The ELEVENTH EDITION, carefully corrected.

To which is added,

A SUPPLEMENT, containing the History of LOGARITHMS, and an INDEX to the whole Work.

DUBLIN: Printed for THOMAS EWING, in Ca[illegible] WILLIAM SMITH, in Dame-street, Booksellers, 17[illegible]

To the HONOURABLE

Sir RICHARD GROSVENOR, of *Eaton* in the County *Palatinate* of *Chester*, Baronet.

SIR,

WHEN requested by some Booksellers in *London*, to Revise and Prepare this Treatise for a New Impression, and once resolved to answer their Demands; I was not long considering at whose Feet to lay it.

My Memory may indeed be impaired by Age, Misfortunes, and Accidents; nay, I am sensible it is so: But it must be entirely lost, when I am forgetful of the great Obligations I lie under to Sir *Richard Grosvenor*.

Your Hospitality and Generosity make you stand unenvied in the Abundance of Fortune. Any Upstart may contrive to spend a great Estate; but it is a Felicity almost peculiar to great Birth to become One.

Were I now to describe Liberality, without Profuseness; Steadiness in Principles, without any private View; Candour and Affability, Good Nature joined to sound Judgment, and a Serenity of Temper, which your Enemies will always find the Companion of true Courage; and then pronounce that you are possessed of all these good Qualities in as high a Degree as most Men living: No Gentleman that knows you well, would think I flattered you.

Sir, Give me Leave to say, I honour your Character, and love your Person; My Expressions are uncourtly,

my Stile unpolished, and therefore more proper to be prefixed to a Work wherein the Matters related are indeed clad in a plain and homely Dress; but they are true, and designed to propogate Mathematical Learning among such as desire to be introduced into that Sort of Knowledge; and I am extremely pleased they are permitted to be sent into the World under your Protection.

That you may long live, to promote the Good of your Country, and that City in whose Interest you have so heartily engaged Yourself; and that you may ever succeed in your own private Affairs, and live to enjoy all the Blessings that attend a quiet prudent Life, is the earnest Prayer of,

Honoured SIR,

Your most obliged, humble,

and obedient Servant,

J. WARD.

To

To the READER.

I *Think it needleſs* (and almoſt endleſs) *to run over all the Uſefulneſs, and Advantages of* Mathematicks *in General; and ſhall therefore only touch upon thoſe two admirable Sciences,* Arithmetick *and* Geometry; *which are indeed the two grand Pillars* (or rather the Foundations) *upon which all other Parts of Mathematical Learning depend.*

As to the Uſefulneſs of Arithmetick, *it is well known that no Buſineſs, Commerce, Trade, or Employment whatſoever, even from the Merchant to the Shop-keeper,* &c. *can be managed and carried on, without the Aſſiſtance of Numbers.*

And as to the Uſefulneſs of Geometry, *it is as certain, that no curious Art, or Mechanick-Work, can either be invented, improved, or performed, without it's aſſiſting Principles; though perhaps the Artiſt, or Workman, has but little* (nay, ſcarce any) *Knowledge in* Geometry.

Then, as to the Advantages that ariſe from both theſe Noble Sciences, when duly joined together, to aſſiſt each other, and then apply'd to Practice, (according as Occaſion requires) *they will readily be granted by all who conſider the vaſt Advantages that accrue to Mankind from the Buſineſs of Navigation only. As alſo from that of Surveying and Dividing of Lands betwixt Party and Party. Beſides the great Pleaſure and Uſe there is from Time-keepers, as Dials, Clocks, Watches, &c. All theſe, and a great many more very uſeful Arts,* (too many to be enumerated here) *wholly depend upon the aforeſaid Sciences.*

And therefore it is no Wonder, that in all Ages ſo many ingenious and learned Perſons have employed themſelves in writing upon the Subject of Mathematicks; *but then moſt of thoſe Authors ſeem to preſuppoſe, that their Readers had made ſome Progreſs in that Sort of Learning before they attempted to peruſe thoſe Books, which are generally large Volumes, written in ſuch abſtruſe Terms, that young Learners were really afraid of looking into thoſe Studies.*

Theſe Conſiderations firſt put me (many Years ago) *upon the Thoughts of endeavouring to compoſe ſuch a plain and familiar Introduction to the* Mathematicks, *as might encourage thoſe that were willing* (to ſpend ſome Time that Way) *to venture and proceed on with Chearfulneſs; though perhaps they were wholly ignorant of its firſt Rudiments. Therefore I began with thoſe firſt Elements or Principles.*

The

is, I began with an Unit in Arithmetick, and a Point in [*Geome*]*try; and from these Foundations proceeded gradually on, leading the young Learner Step by Step with all the Plainness I could,* &c.

And for that Reason I published this Treatise (Anno 1707) *by the Title of the* Young Mathematician's Guide; *which has answered the Title so well, that I believe I may truly say* (without Vanity) *this Treatise hath proved a very helpful Guide to near five thousand Persons; and perhaps most of them such as would never have looked into the* Mathematicks *at all but for it.*

And not only so, but it hath been very well received amongst the Learned, and (I have been often told) *so well approved of at the Universities, in* England, Scotland, *and* Ireland, *that it is ordered to be publickly read to their Pupils,* &c.

The Title Page gives a short Account of the several Parts treated of, with the Corrections and Additions that are made to this Fifth Edition, *which I shall not enlarge upon, but leave the* Book *to speak for itself; and if it be not able to give Satisfaction to the Reader, I am sure all I can say here in its Behalf will never recommend it: But this may be truly said, That whoever reads it over, will find more in it than the Title doth promise, or perhaps he expects: it is true indeed, the Dress is but Plain and Homely, it being wholly intended to instruct, and not to amuse or puzzle the young Learner with hard Words, and obscure Terms: However, in this I shall always have the Satisfaction; That I have sincerely aimed at what is useful, though in one of the meanest Ways; it is Honour enough for me to be accounted as one of the Under-Labourers in clearing the Ground a little, and removing some of the Rubbish that lay in the Way to this Sort of Knowledge. How well I have performed That, must be left to proper Judges.*

To be brief; as I am not sensible of any Fundamental Error in this Treatise, so I will not pretend to say it is without Imperfections, (Humanum est errare) *which I hope the Reader will excuse, and pass over with the like Candour and Good-Will that it was composed for his Use; by his real Well-wisher,*

J. WARD.

London, October 10th, 1706.

Corrected, *&c.* at Chester,
January 20th, 1722.

THE

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A N

AN INTRODUCTION TO THE *MATHEMATICKS.*

PART I.

PRÆCOGNITA.

THE *Business of* Mathematicks, *in all its Parts, both* Theory *and* Practice, *is only to search out and determine the true Quantity, either of* Matter, Space, *or* Motion, *according as Occasion requires.*

By Quantity *of* Matter *is here meant the* Magnitude *or Bigness of any visible Thing, whose* Length, Breadth, *and* Thickness, *may either be measured, or estimated.*

By Quantity *of* Space *is meant the Distance of one Thing from another.*

And by Quantity *of* Motion *is meant the Swiftness of any Thing moving from one Place to another.*

The Consideration of these, according as they may be proposed, are the Subjects *of the* Mathematicks, *but chiefly that of* Matter.

Now the Consideration of Matter, *with respect to its* Quantity, Form *and* Position, *which may either be* Natural, Accidental, *or* Designed, *will admit of infinite Varieties: But all the Varieties that are yet known, or indeed possible to be conceived, are wholly comprized under the due Consideration of these Two,* Magnitude *and* Number, *which are the proper Subjects of* Geometry, Arithmetick, *and* Algebra. *All other* Parts *of the* Mathematicks *being only the Branches of these three* Sciences, *or rather their Application to particular Cases.*

GEOMETRY *is a* Science *by which we search out, and come to know, either the whole* Magnitude, *or some Part of any proposed* Quantity; *and is to be obtained by comparing it with another known Quantity of the same Kind, which will always be one of these,* viz. *A* LINE, *(or* Length *only) A* SURFACE, *(that is,* Length *and* Breadth) *or a* SOLID, *(which hath* Length, Breadth, *and* Depth, *or* Thickness) *Nature admitting of no other Dimensions but these Three.*

ARITHMETICK *is a* Science *by which we come to know what* Number *of* Quantities *there are (either real or imaginary) of any Kind, contained in another* Quantity *of the same Kind: Now this Consideration is very different from that of* Geometry, *which is only to find out true and proper Answers to all such Questions as demand, how* Long, *how* Broad, *how* Big, *&c. But when we consider either more* Quantities *than one, or how often one* Quantity *is contained in another, then we have Recourse to* Arithmetick, *which is to find out true and proper Answers to all such Questions as demand, how* Many, *what* Number, *or* Multitude *of* Quantities *there are. To be brief, the Subject of* Geometry *is that of* Quantity, *with respect to its* Magnitude *only; and the Subject of* Arithmetick *is* Quantities *with respect to their* Number *only.*

ALGEBRA *is a* Science *by which the most abstruse or difficult Problems, either in* Arithmetick *or* Geometry, *are Resolved and Demonstrated; that is, it equally interferes with them both; and therefore it is promiscuously named, being sometimes called* Specious Arithmetick, *as by* Harriot, Vieta, *and Dr.* Wallis, *&c. And sometimes it is called* Modern Geometry, *particularly the ingenious and great Mathematician Dr.* Edmund Halley, *Savilian Professor of* Geometry *in the University of* Oxford, *and Royal Astronomer at* Greenwich, *giving this following Instance of the Excellence of our* Modern Algebra, *writes thus.*

'*The Excellence of the* Modern Geometry *(saith he) is in nothing more evident, than in those full and* Adequate Solutions *it gives to* Problems; *representing all the possible Cases at one View, and in one general* Theorem *many Times comprehending whole* Sciences; *which deduced at length into* Propositions, *and demonstrated after the Manner of the* Antients, *might well become the Subjects of large Treatises: For whatsoever* Theorem *solves the most complicated* Problem *of the Kind, does with a due Reduction reach all the subordinate* Cases.' *Of which he gives a notable Instance in the* Doctrine *of* Dioptricks *for finding the* Foci *of* Optic Glasses *universally. (Vide* Philosophical Transactions, *Numb.* 205.)

Thus

Thus you have a short and general Account of the proper Subjects of those three noble and useful Sciences, Arithmetick, Geometry *and* Algebra. *I shall now proceed to give a particular Account of each; and first of* Arithmetick, *which is the* Basis *or* Foundation *of all* Arts, *both* Mathematick *and* Mechanick; *and therefore it ought to be well understood before the rest are meddled withal.*

CHAP. I.

Concerning the several Parts of Arithmetick, *with the Definition of such Characters as are used in this Treatise.*

ARITHMETICK, or the *Art* of *Numbering*, is fitly divided into three distinct Parts, two of which are properly called *Natural*, and the third *Artificial*.

The first, being the most plain and easy, is commonly called *Vulgar Arithmetick* in whole *Numbers*; because every *Unit* or *Integer* concerned in it, represents one whole *Quantity* of some *Species* or Thing proposed.

The second is that which supposes an *Unit* (and consequently the *Quantity* or Thing represented by that *Unit*) to be *Broken* or *Divided* into equal *Parts* (either even or uneven) and considers of them either as pure *Parts*, *viz.* Each less than an *Unit*, or else of *Parts* and *Integers* intermixt. And is usually called the *Doctrine of Vulgar Fractions*.

The third, or *Artificial Part*, is called *Decimal Arithmetick*; being an *Artificial* Invention of managing *Fractions* or *Broken Numbers*, by a much more commodious and easy Way than that of *Vulgar Fractions*: For the several Operations performed in *Decimals*, differ but little from those in *Whole Numbers*: And therefore it is now become of general Use, especially in *Geometrical Computations*.

ARITHMETICK (in all its *Parts*) is performed by the various ordering and disposing of Ten *Arabick Characters* or *Numerical Figures* (which by some are called *Digits*.)

viz.	*One*,	*Two*,	*Three*,	*Four*,	*Five*,	*Six*,	*Seven*,	*Eight*,	*Nine*,	*Cypher*.
	1	2	3	4	5	6	7	8	9	0

The Use of these Characters *is said to be first introduced into* England *near six hundred Years ago,* viz. *about the Year* 1130, *vide Dr.* Wallis's *Algebra*, Page 12.

The firſt of theſe *Characters* is called *Unity*, and repreſents one of any Kind of *Species* or *Quantity*. As one *World*, one *Star*, one *Man*, &c.

Viz. *Unity* is that by which every Thing that is, is called one, (*Euclid* 7. *Def.* 1.) and is the Beginning of all *Numbers*. That is to ſay, *Number* is a *Multitude* of *Units*. Euclid 7. Def. 2.

For, one more one, makes Two; and one, more one, more one, makes Three, &c. *Which is the firſt and chief* Poſtulate, or *rather* Axiom *to* Arithmetick.

Viz. That $1+1=2$. $1+1+1=3$. $1+1+1+1=4$. $1+1+1+1+1=5$. And ſo on to 9.

Nine of theſe *Figures* were thus compoſed of *Units*, and differently form'd to repreſent ſo many *Units* put together into one *Sum*, as was intended each ſhould denote: *Nine* being the greateſt *Number* of *Units* that was then thought convenient to be expreſſed by one ſingle *Character*; the laſt of the *Ten* is only a *Cypher*, or (as ſome phraſe it) a Nothing, becauſe of itſelf it ſignifies nothing; for if never ſo many *Cyphers* be Added to, or Subſtracted from any *Number*, they can neither increaſe nor diminiſh that *Number*; but yet, as a *Cypher* (or *Cyphers*) may be placed, the other *Figures* will become of different *Values* from what they were before, as will appear further on.

For the more convenient ordering of the aforeſaid *Numeral Figures*, according to the ſeveral Varieties that happen in *Computations*; I do adviſe the young Learner to acquaint himſelf with the Signification of the following *Algebraick Signs* or *Characters*, which he will find of excellent Uſe, as being a much ſhorter, better, and more ſignificant Way of denoting what is to be done (in moſt Operations) than can otherwiſe be expreſſed in Words at length.

Significations.

Signs	*Names.*	
$+$	*Plus* or *more.*	The Sign of *Addition*; as $8+7$ is 8 more 7, and ſignifies that the *Numbers* 8 and 7 are to be added into one *Sum*. The like is to be underſtood when ſeveral *Numbers* are connected together with the Sign $+$. As $34+22+9+45$, &c. denotes theſe are all to be added into one *Sum*.

The

—	*Minus or less.*	The Sign of *Substraction*; as 9—6, is 9 less 6, and signifies that 6 is to be taken from 9, that so their Difference may be found.
×	*Into or with.*	The Sign of *Multiplication*; as 9×6, is 9 into 6, and signifies that 9 is to be multiplied into or with 6.
÷	*By*	The Sign of *Division*; as 8÷2, is 8 by 2, and signifies that 8 is to be Divided by 2, also thus 2)8(4, or thus $\frac{8}{2}$, each signifying the same Thing, to wit, 8 Divided by 2.
=	*Equal.*	The Sign of *Equality* or *Equation*, viz. whenever this Sign = is placed betwixt *Numbers* (or *Quantities*) it denotes them to be Equal, as 9=9, or 9+6=15, or 9—6=3, *&c.* That is, 9 is Equal to 9, or 9 more 6 is Equal to 15, and 9 less 6 is Equal to 3, *&c.*
: :	*So is.*	The Sign of *Proportion*, or that commonly called the *Golden Rule*, or *Rule of Three*, and : : is always placed betwixt the Two middle *Terms* or *Numbers* in Proportion. Thus 2 : 8 : : 6 : 24. To be read thus; as 2, is to 8, so is 6, to 24.

These *Signs* and their *Significations*, being perfectly learnt, will help to shorten the Work.

CHAP. II.

Concerning the Principal Rules *in* ARITHMETICK, *and how they are performed in Whole Numbers.*

THE *Rules* by which *Numerical Operations* are perform'd in all the Parts of *Arithmetick*, are many and various, several of them being form'd and raised as Occasion requires, when applied to *Practice*; yet they are all comprehended within the due Consideration of these Six, *viz.* NUMERATION (or NOTATION)

TION) ADDITION, SUBSTRACTION, MULTIPLICATION, DIVISION, and EVOLUTION, or *Extraction of Roots.*

Sect. 1. *Of* NUMERATION *or* NOTATION.

NUMERATION or *Notation*, teacheth to read or express the true Value of any *Number* when writ down; and consequently to write down any proposed *Number* according to its true Value when it is named: and this consisteth of Two Parts.

1. The due Order of placing down Figures.
2. The true valuing of each Figure in its Place.

Both which are plainly exhibited in the following Table.

&c.	*Hundreds of Thousands of Millions*	*Tens of Thousands of Millions*	*Thousands of Millions*	*Hundreds of Millions*	*Tens of Millions*	*Millions*	*Hundreds of Thousands*	*Tens of Thousands*	*Thousands*	*Hundreds*	*Tens*	*Units* } *And is the first Place in Counting*
	6	7	8	9	8	7	6	5	4	3	2	1
	Period of Thousands of Millions.			*Period of Millions.*			*Period of Thousands.*			*Period of Units.*		

By this NUMERATION *Table* it is apparent, that the Order of Places is reckoned from the Right-hand towards the Left; the first Place of any *Number* being always that which is the outmost *Figure* to the Right-hand: And whatever *Figure stands* in that Place, doth only signify its own simple Value, *viz.* so many *Units* as that *Figure* represents.

The second Place is that of *Tens*, and any *Figure* standing in that Place signifieth so many *Tens* as that *Figure* represents *Units*.

The

The third Place is *Hundreds*, the fourth Place *Thousands*, &c. That is, each Place towards the Left-hand is *Ten* Times the Value of that next it, towards the Right.

For Inſtance, ſuppoſe 759 were propoſed to be read or pronounced according to the Value of each *Figure* as they now ſtand. The firſt *Figure* in this *Sum* is 9, becauſe it ſtands in the Place of *Units*, and therefore ſignifies but its own ſimple Value, to wit, 9 *Units*, or 9. The ſecond *Figure* 5 ſtands in the Place of *Tens*, and therefore ſignifies five *Tens* or *Fifty*. The *Figure* 7 ſtands in the third Place, or Place of *Hundreds*, and therefore it ſignifies *Seven Hundred*; and the whole Sum is to be read or pronounced thus, *Seven Hundred Fifty Nine.*

Note, Altho' the *Figure* 7 ſtands in the third Place (according to the Order of *Numbering)* yet when the whole *Sum* comes to be read, it is firſt pronounced; the reading of *Numbers* being performed like that of Letters or Words, always beginning with the outmoſt *Figure* towards the Left-hand, and ſo many *Figures* as are placed together without any Point, Comma, Line, or other Note of Diſtinction between them, are all but one *Sum*, and muſt be read as ſuch.

For Example, 763596 is but one entire *Sum* or *Number*, notwithſtanding it conſiſts of ſix Places of *Figures*, and is thus read; *Seven Hundred Sixty Three Thouſand, Five Hundred Ninety Six.*

The like is to be obſerved in reading or expreſſing the true Value of any *Sum* or *Rank* of *Numbers* conſiſting of *Seven*, *Eight*, *Nine*, or more Places of *Figures*, each *Figure* being to be valued according to its Diſtance from the Place of *Unity:* As in the foregoing Table.

Now ſuch Values may as well ariſe by *Cyphers*, as by other *Figures*; for Inſtance, 6 ſtanding by itſelf, repreſents but Six *Units:* But if a *Cypher* be annext to it thus, 60, then it becomes *Sixty*; for the *Cypher* poſſeſſing the Place of *Units*, hath thereby removed the 6 into the Place of *Tens*; and another *Cypher* more would make it 600, *Six hundred*, &c.

Whence it may be noted, that although a *Cypher* of itſelf ſignify nothing (as hath been ſaid before) yet being placed on the Right-hand of any *Figure*, it augments the Value of that *Figure* by advancing it into a higher Place than otherwiſe it would have been, had not the *Cypher* been there.

Take one Example more in *Numeration* (if you pleaſe, that in the Table) *viz.* 67898654321, which is, according as is there ſignified.

Six

Six Hundred Seventy Eight Thousand Millions,
Nine Hundred Eighty Seven Millions,
Six Hundred Fifty Four Thousand,
Three Hundred Twenty One Units. Of any proposed *Species* or *Quantities* whatsoever.

And here it may be observed, that every third *Figure* from the Place of *Units*, bears the Name of *Hundreds*; which shews that if any great *Sum* be parted, or rather distinguished into Periods, of Three *Figures* in each *Period* (as in the foregoing Table) it will be of good Use to help the young Learner in the easier valuing and expressing that *Sum*.

Sect. 2. *Of* Addition.

Postulate or *Petition.*

That any given Number *may be increased or made more, by putting another* Number *to it.*

Addition is that *Rule* by which several *Numbers* are collected and put together, that so their *Sum* or *Total* Amount may be known.

In this *Rule* two Things being carefully observed, the Work will be easily performed.

1. The first is the true placing of the *Numbers*, so as that each *Figure* may stand directly underneath those *Figures* of the same Value, *viz.* place *Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c.

Then underneath the lowest Rank (always) draw a Line to separate the given *Numbers* from their *Sum* when it is found.

Example. If these *Numbers* 54327, and 2651, were given to be added together, they must be placed

$$\text{Thus,}\ \left\{ \begin{array}{r} 54327 \\ 2651 \\ \hline \end{array} \right.$$

2. The second Thing to be observed is the due Collecting or Adding together each Row of *Figures* that stand over one another of the same Value: And that is thus performed.

Rule.

Always begin your Addition *at the Place of* Units, *and Add together all the* Figures *that stand in that Place, and if their* Sum *be under* Ten, *set it down below the Line underneath its own Place; but if their Sum be more than* Ten, *you must set down only the Overplus, or odd* Figure *above the* Ten *(or* Tens) *and so many* Tens *as the* Sum *of these* Units *amount to, you must carry*

to

to the Place of Tens; *Adding them and all the* Figures *that stand in the Place of* Tens *together, in the same Manner as those of the* Units *were added; then proceed in the same Order to the Place of* Hundreds, *and so on to each Place until all is done.*

The *Sum* arising from those *Additions* will be the *Total* Amount required.

EXAMPLE 1.

Let it be required to find the Sum of the aforesaid *Numbers*,

viz. { 54327
2651 }

56978 the *Sum* required.

Beginning at the Place of *Units*, I say 1 and 7 is 8, which being less than 10, I set it down (according to the *Rule)* underneath its own Place of *Units*; and then proceed to the Place of *Tens*, saying 5 and 2 is 7, which being less than 10, I set it down underneath its own Place of *Tens*, and proceed to do the like at the Place of *Hundreds*, and then at *Thousands*, setting each of their *Sums* underneath their own respective Places: Lastly, because there is not any *Figure* in the lower Rank to be added to the *Figure* 5, which stands in the Place of *Ten Thousands*, in the upper Rank, I therefore bring down the said 5 to the rest, placing it underneath its own Place, and then I find that 54327+2651=56978, is the true *Sum* required.

EXAMPLE 2.

Suppose it were required to find the *Sum* of these *Numbers*, 3578+496+742+184+95. These being placed, as before directed, will stand as in the Margin. Then beginning (as before) at the Place of *Units*, say 5 and 4 is 9, and 2 is 11, and 6 is 17, and 8 is 25; set down the 5 *Units* underneath its own Place of *Units*, and carry the 20, or two *Tens*, to the Place of *Tens* (at which Place they are only 2) saying, 2 and 9 is 11, and 8 is 19, and 4 is 23, and 9 is 32, and 7 is 39; set down the 9 underneath its own Place of *Tens*, and carry the 30, or three *Tens* (which indeed is 300) to the Place of *Hundreds*, at which Place they are but 3, saying, 3 I carry and 1 is 4, and 7 is 11, and 4 is 15, and 5 is 20; here because there is no *Figure* overplus (as before) I set down a *Cypher* underneath the Place of *Hundreds*, and carry the 2 *Tens* (or rather the 2000) to the Place of *Thousands*, saying

3578
496
742
184
95
——
5095

(as before) 2 I carry and three is 5, which being the last, I set it down underneath its own Place, and all is finished. And find the *Sum* or *Total* amount to be $5095 = 3578 + 496 + 742 + 184 + 95$.

If this Example be well considered, it will be sufficient to shew the usual Method of *Addition* in whole *Numbers*; but to make all plain and clear, I shall shew the young Learner the Reason of carrying the *Tens* from one Degree or Row of *Figures*, to the next superior Degree, which is done purely to save Trouble, and prevent the using of more *Figures* than are really necessary, as will appear by the following Method of adding together the same *Numbers* of the last Example.

Thus, add together each single Row of Figures by itself; as if there were no more but that one Row, setting down the *Sum* underneath its own Place.

3	5	7	8
	4	9	6
	7	4	2
	1	8	4
		9	5

The *Sum* of the Row of *Units*, is			2	5	Ad
The *Sum* of the Row of *Tens*, is		3	7	0	
The *Sum* of the Row of *Hund.* is	1	7	0	0	
The three *Thousand* brought down	3	0	0	0	
The *Sum* or *Total* Amount as before, is	5	0	9	5	

From hence I presume it will be easy to conceive the true Reason of carrying the aforesaid *Tens*; and also that *Cyphers* do not augment or increase the *Sum* in *Addition*. (See Page 4.)

I might have here inserted a Lineal Demonstration of this *Rule* of *Addition*; but I thought it would rather puzzle than improve a young Learner, especially in this Place; besides the Reason of it is sufficiently evident from that natural Truth of *the Whole being Equal to all its Parts taken together.* Euclid 1. Axiom 19.

That is, the *Numbers* which are proposed to be added together, are by that *Axiom* understood to be the several Parts, and their *Sum* or *Total* Amount found by *Addition* is understood to be the Whole.

And from thence is deduced the Method of proving the Truth of any *Operation* in *Addition*, viz. By parting or separating the given *Numbers* into two Parcels (or more, according to the Largeness of it) and then adding up each Parcel by itself: For if those particular *Sums* so found, be *added* into one *Sum*, and that *Sum* prove Equal, or the same with the *Total Sum* first found,

found, then all is right; if not, Care must be taken to discover and correct the Error.

EXAMPLE.

Add	5647 3289 4016	The *Sum* of these Parts is,	12952
	2900 5007 1606	The *Sum* of these, is	9513
The *Total Sum* of all these Parts	22465	The *Sum* of each Parcel put together	22465

Sect. 3. Of Substraction.

Postulate or *Petition.*

That any NUMBER *may be diminished, or made less, by taking another* NUMBER *from it.*

SUBSTRACTION is that *Rule* by which one *Number* is *deducted* or taken out of another, that so the *Remainder, Difference,* or *Excess* may be known.

As 6 taken out of 9, there *remains* 3. This 3 is also the *Difference* betwixt 6 and 9, or it is the *Excess* of 9 above 6.

Therefore the *Number* (or *Sum*) out of which *Substraction* is required to be made, must be greater than (or at least equal to) the *Subtrahend* or *Number to be substracted.*

Note, *This Rule is the Converse or direct Contrary to* Addition.

And here the same Caution that was given in *Addition,* of placing *Figures* directly under those of the same Value, *viz. Units* under *Units, Tens* under *Tens,* and *Hundreds* under *Hundreds,* &c. must be carefully observed; also underneath the lowest Rank there must be drawn a Line (as before in *Addition*) to separate the given *Numbers* from their *Difference* when it is found.

Then having placed the lesser *Number* under the greater, the *Operation* may be thus performed.

RULE.

Begin at the Right-hand Figure *or Place of* Units *(as in* Addition) *and take or substract the lower* Figure *in that Place*

from the Figure *that ſtands over it, ſetting down the* Remainder *or* Difference *underneath its own Place. If the Two* Figures *chance to be Equal, ſet down a* Cypher : *But if the upper* Figure *be leſs than the lower* Figure, *then you muſt add* 10 *to the upper* Figure, *or mentally call it* 10 *more than it is, and from that* Sum ſubſtract *the lower* Figure, *ſetting down the* Remainder *(as* before directed.) *Now becauſe the* 10 *thus added, was ſuppos'd to be borrowed from the next ſuperior Place* (viz. *of* Tens) *in the upper* Figures, *therefore you muſt either call the upper* Figure *in that Place from whence the* 10 *was borrowed, one leſs than really it is, or elſe (which is all one, and moſt uſual) you muſt call the lower* Figure *in that Place one more than it really is, and then proceed to* Subſtraction *in that Place, as in the former ; and ſo gradually on from one Row of* Figures *to another until all be done.*

EXAMPLE 1.

Let it be required to find the *Difference* between 6785, and 4572. That is, let 4572 be *ſubſtracted* from 6785.

Theſe *Numbers* being placed down, as before directed, will ſtand

$$\text{Thus}\ \left\{\begin{array}{r} 6785 \\ \underline{4572} \\ 2213 \end{array}\right.$$

Beginning at the Place of *Units*, take 2 from 5 and there will *remain* 3 which muſt be ſet down underneath its own Place, and then proceed to the Place of *Tens*, taking 7 from 8, and there will *remain* 1, to be ſet down underneath its own Place; again, at the Place of *Hundreds*, take 5 from 7, and there *remains* 2, which ſet down, as before; laſtly, take 4 from 6 and there will *remain* 2, which being ſet down underneath its own Place, the Work is finiſhed, and the *Difference* ſo found will be 2213=6785—4572, as was required.

EXAMPLE 2.

The *Difference* between 5849, and 7496 is required.

Having placed the *Numbers* as in the Margin, begin at the Place of *Units* (as before) and ſay 9 from 6 cannot be, but 9 from 16 and there *remains* 7, to be ſet down under its own Place; next proceed to the Place of *Tens*, where you muſt now pay the 10 that was borrowed to make the 6, 16, by accounting the upper *Figure* 9 in that Place one leſs than it is, ſaying 4 from 8 and there *remains* 4, or elſe (which is the moſt practiſed) ſay 1 I borrowed and 4 is 5 from

$$\begin{array}{r} 7496 \\ \underline{5849} \\ 1647 \end{array}$$

from 9 and there *remains* 4, to be ſet down under its own Place (as before); again, at the Place of *Hundreds*, ſay 8 from 4 that cannot be, but 8 from 14 there will *remain* 6 to be ſet down; and here I have borrowed 10 (as before) which muſt be paid in the ſame manner as the other 10 was, *viz.* either by calling the 7 in the upper Rank but 6, ſaying 5 from 6 there *remains* 1, or elſe by ſaying 1 borrowed and 5 is 6 from 7 and there *remains* 1, which being ſet down under its own Place all is done, and the *Difference* required will be 1647=7496—5849.

EXAMPLE 3.

From 830476
Take 741068
———
89408

By this Example you may perceive that *Cyphers* in the *Subtrahend*, viz. in the *Numbers* to be *ſubſtracted*, do not diminiſh the *Number* from whence *Subſtraction* is made. See *Page* 4.

Theſe Three Examples, I preſume may be ſufficient to ſhew the young Learner the Method of *ſubſtracting* whole *Numbers*; as for the Reaſon thereof it is the ſame with that of *Addition*, Page 10, *viz.* of *the Whole being equal to all its Parts taken together.*

That is, in this *Rule* the *Number* from which *Subſtraction* is required to be made, is underſtoood to be the Whole, and the *Subtrahend* or *Number* to be *ſubſtracted*, is ſuppos'd to be a Part of that Whole; conſequently if that Part be taken from the Whole, the *Remainder* will be the other Part.

From hence is deduced the common Method of proving *Subſtraction*, by adding together the *Subtrahend* and the *Remainder*. For if the *Sum* of thoſe Two (which are here called Parts) be equal to the *Number* from whence *Subſtraction* was made (which is here called the Whole) then the Work is right; if not, care muſt be taken to diſcover and correct the Error.

EXAMPLE.

From 59435
Take 47608 } Add
———
Proof { 11827
———
59435 } The *Sum* which is equal to the *Number* from whence *Subſtraction* was made.

Or

Or from the abovesaid Reason, it will be easy to conceive how to prove the Truth of *Substraction* by *Substraction.*

For if from	59435	being here the whole,
there be taken	47608	as part of the whole ;
there will *remain*	11827	the other part (as before)
And if from	59435	the whole, there be *substracted* the
last part, *viz.*	11827	
there will *remain*	47608	the first part, or *Number* which was

required to be first *substracted.*

From	75643	From	7000000
Take	9000	Take	986432
Remains	66643	Remains	6013568

Sect. 4. Of Multiplication.

Multiplication is a *Rule* by which any given *Number* may be speedily increased, according to any proposed *Number* of *Times.*

That is, One Number *is said to* Multiply *another, when the* Number multiplied *is so often added to itself, as there are* Units *in the* Number multiplying ; *and another* Number *is produced,* (Euclid 7. Def. 15.)

To perform *Multiplication,* there is required two given *Numbers,* called *Factors.*

The First is the *Number* to be *multiplied,* which is generally put the greater of the Two *Numbers,* and is commonly call'd the *Multiplicand.*

The other is that *Number* by which the first is to be *multiplied,* and is usually called the *Multiplicator* or *Multiplier* , and this denotes the *Number* of *Times* that the *Multiplicand* is required to be *added* to itself. For so many *Units* as are contained in the *Multiplier,* so many times will the *Multiplicand* be really *added* to itself *(as per* Euclid *above).* And from thence will arise a Third *Number,* called the *Product.* But in *Geometrical Operations* it is called the *Rectangle* or *Plain.*

For Instance ; suppose it were required to increase 6 four times, that is, to *multiply* 6 into or with 4. These Two *Numbers* are to be set (or placed) down as in *Addition* or *Substraction.*

Thus

Thus $\left\{\begin{array}{ll} 6 & \textit{Multiplicand,} \\ 4 & \textit{Multiplier,} \end{array}\right\}$ or *Factors.*

Product 24 *viz.* 4 times 6 is 24, as plainly appears by *Addition, viz.* By setting down 6 four times, and then adding them together into one *Sum,* Thus

$$\left.\begin{array}{l|l} 1 & 6 \\ 2 & 6 \\ 3 & 6 \\ 4 & 6 \end{array}\right\} \text{Add}$$

$$\overline{\quad 24 \quad}$$

From hence it is evident that Multiplication *is only a* Concise *or* Compendious *Way* of adding *any given* Number *to itself, so often as any* Number *of Times may be proposed.*

Before any Operation can be readily performed in *Multiplication,* the several *Products* of the single *Figures* one into another must be perfectly learn'd by Heart, *viz.* That 2 times 2 is 4, that 3 times 3 is 9, and 3 times 6 is 18, &c. According as they are expressed in the following *Table*; wherein I have omitted *multiplying* with 2, it being so very easy that any one may do it.

Multiplication Table.

$3\times3=9$	$4\times4=16$	$5\times5=25$	$6\times6=36$	$7\times7=49$	$8\times8=64$
$3\times4=12$	$4\times5=20$	$5\times6=30$	$6\times7=42$	$7\times8=56$	$8\times9=72$
$3\times5=15$	$4\times6=24$	$5\times7=35$	$6\times8=48$	$7\times9=63$	$9\times9=81$
$3\times6=18$	$4\times7=28$	$5\times8=40$	$6\times9=54$		
$3\times7=21$	$4\times8=32$	$5\times9=45$			
$3\times8=24$	$4\times9=36$				
$3\times9=27$					

I think it needless to give any *Explanation* of this *Table*; for if the *Signs* and their *Significations* be well understood (*vide Page* 5) it must needs be easy. Only this may be noted, that $4\times3=3\times4$, or $7\times5=5\times7$, &c.

That is, 3 times 4 is the same with 4 times 3, or 5 times 7 is the same with 7 times 5, &c. The like must be understood of all the rest in the *Table.*

And when all those single *Products* are so perfectly learn'd by Heart, as to be said without pausing; you may then proceed (but not till then) to the Business of *Multiplication*; which will be found very easy, if the following *Rule* (and *Examples*) be carefully observed.

RULE.

Always begin with that Figure *which stands in the* Units *Place of the* Multiplier, *and with it* multiply *the* Figure *which stands*

in

in the Units *Place of the* Multiplicand; *if their* Product *be less than Ten, set it down underneath its own Place of* Units, *and proceed to the next* Figure *of the* Multiplicand. *But if their* Product *be above* Ten *(or* Tens) *then set down the Overplus only (or odd* Figure, *as in* Addition) *and bear (or carry) the said* Ten *or* Tens *in Mind until you have* multiplied *the next* Figure *of the* Multiplicand, *with the same* Figure *of the* Multiplier; *then to their* Product *add the* Ten *or* Tens *carried in Mind, setting down the Overplus of their* Sum *above the* Tens, *as before*; *and so proceed on in the very same Manner, until all the* Figures *of the* Multiplicand *are* multiplied *with that* Figure *of the* Multiplier.

EXAMPLE 1.

Suppose it were required to *multiply* 3213 into or with 3.

3213	*Multiplicand,*	or *Factors.*
3	*Multiplier,*	
Product 9639		

Beginning at the *Units* Place, say 3 times 3 is 9, which, because it is less than *Ten*, set down underneath its own Place, and proceed to the next Place of *Tens*, saying 3 times 1 is 3, which set down underneath its own Place; then to the next Place, *viz.* of *Hundreds*, saying 3 times 2 is 6, which set down, as before; lastly, at the Place of *Thousands*, say 3 times 3 is 9, which being set down underneath its own place, the Operation is finished; and the true *Product* is $9639=3213\times3$, as was required.

EXAMPLE 2.

Let it be required to *multiply* 8569 into 8. Set down these *Numbers* as before.

Thus { 8569
8

68552

Beginning at the *Units* Place, say 8 times 9 is 72, set down the 2 underneath its own Place of *Units*, and bear the 70, or 7 *Tens* in Mind, and proceed to the next *Figure* of the *Multiplicand*, (at which Place the 7 *Tens* are only 7) saying 8 times 6 is 48, and the 7 carried in Mind is 55; set down the odd 5 underneath its own Place of *Tens*, and carry the 50 (which is really 500) to the next Place *(viz.* of *Hundreds)* at which Place it is only 5, where say, 8 times 5 is 40, and the 5 carried in Mind is 45; set down the 5 underneath its own Place, and carry the 40 or 4 *Tens* (which is really 4000) to the next

next Place, *viz.* of *Thousands*, saying, 8 times 8 is 64, and 4 carried in Mind is 68. (Now this being the last Place or *Figure* to be *multiplied)* set down the whole *Product* 68, and the Work is done.

So that, 8569×8=68552, the *Product* required.

Now the Reason of this and all other the like Operations, may be easily conceived from this which follows.

8569 8	The same *Factors* as before
72	Here 8 times 9 is 72, as before, because the 9 stands in the *Units* Place.
480	Now here it is not really 8 times 6=48, but it is 8 times 60=480, because the 6 stands in the Place of *Tens.*
4000	And here it is not 8 times 5=40, but it is really 8 times 500=4000, because the 5 stands in the Place of *Hundreds.*
64000	Lastly, because the 8 in the *Multiplicand* stands in the Place of the *Thousands*, it is therefore 8 times 8000=64000, and not 8 times 8=64.
68552	The *Sum* of the particular *Products*, which gives the true *Product*, as before.

By what hath been already said, with a little Consideration had to the *Examples*, I presume the Learner may easily understand how to *multiply* whole *Numbers* with any single *Figure*. And when it is requir'd to *multiply* with more than one; then so many Figures as there are in the *Multiplier*, so many particular *Products* there must be.

That is, all the *Figures* of the *Multiplicand* must be *multiplied* with every single *Figure* of the *Multiplier*, as if there were but one single *Figure:* and the *Sum* of all those particular *Products*, will be the true *Product* required. But in those Operations, great Care must be taken in setting down the particular *Products* (which arise by each *multiplying Figure)* in their proper Places. Which will be easily done, if the following Directions be carefully observed.

Viz. { Always place the first *Figure* (or *Cypher)* of every particular *Product*, directly underneath the *multiplying Figure*. Or thus:

The First Figure *(or* Cypher) *of the second particular* Product *must stand directly under the second* Figure *(or Place) of the First* Product; *and the First* Figure *(or* Cypher) *of the Third*

 particular

particular Product, *must stand directly underneath the Third* Figure *of the First* Product: *And so on until all is done.*

Now the Reason of placing the first *Figure* of every particular *Product* in this Order, will be very obvious to any one that considers the last *Example*; wherein the *Cyphers* are only set down to shew the true Distance of the first *Figure* in each particular *Product* from the *Units* Place. And altho' it is not usual to set down *Cyphers* in this Manner; yet they are always suppos'd to be there: That is, their Places are always left void, as in the two following *Examples*; wherein I have placed *Points* instead of *Cyphers*.

EXAMPLE 3.

Let it be required to *multiply* 78094, into or with 7563.

78094	*Factors.*	
7563		
234282	The First particular *Product* with	3
468564 .	The Second particular *Product* with	60
390470 . .	The Third particular *Product* with	500
546658 . . .	The Fourth particular *Product* with	7000
590624922	The *Total*, or true *Product* required.	

EXAMPLE 4.

Suppose it be required to *multiply* 57498 into 60008.

57498		
60008		
459984	The *Product* with	8
344988	The *Product* with	60000

345039984=57498×60008, as was required.

Here you may observe, that I pass over the *Cyphers*, and only take Care of placing the first *Product* of the last *Figure*, viz. of 60000 according to the foregoing Directions.

When there is a *Cypher* or *Cyphers*, to the Right-hand either of the *Multiplicand* or *Multiplicator*, or to both; in that Case *multiply* the *Figures* as before; neglecting the *Cyphers* until the particular *Products* are added together; then to their *Sum* annex so many *Cyphers* as are in either or both *Factors*. As in these:

EXAMPLE

EXAMPLE 5.	EXAMPLE 6.	EXAMPLE 7.
9538	87600	785000
4600	79	56900
57228	7884	7065
38152	6132	4710
		3925
43874800	6920400	44666500000

Take a few Examples without their Work at large.

75649×579=43800771
687000×356=244572000
530674×45007=23884044718
7901375×30000=237041250000
537084000×590700=317255518800000
102030405×504030201=51426405540261405
987654321×123456789=121932631112635269

Note, If it be required to *multiply* any *Number* with 10, 100, 1000, 10000, &c. it is only annexing the *Cyphers* of the *Multiplier* to the *Figures* of the *Multiplicand*, and the Work is done.

Thus { 578×10 =5780. 578×1000 =578000
578×100=57800 578×10000=5780000, &c.

These *Examples* (being well understood) are sufficient to instruct the Learner in all the Varieties that can happen in *multiplying* of whole *Numbers*, according to the Method generally practised: However, it may not be amiss to shew here how *Multiplication* may be performed (with many *Figures*) by *Addition* only.

EXAMPLE.

Let it be required to *multiply* 879654 into 79863.

In order to perform this (or any other Operation of this kind) by *Addition* only; you must make a Tariffa or small Table of the given *Multiplicand*, in this Manner:

First, Make a small Column, and in it place gradually downward the nine single *Figures*, viz. 1, 2, 3, 4, 5, &c.

 Then

Then against the *Figure* 1, set down the *Multiplicand* (which in this *Example* is 879654) and against the *Figure* 2, set down the Double of the *Multiplicand*, found by *adding* it to itself: To this Double *add* the *Multiplicand*, setting down their *Sum* against the *Figure* 3. And so proceed on by a continued *Addition* until there be ten Times the *Multiplicand* in the *Table*; which if the Work is true, will be the *Multiplicand* itself with a *Cypher* to the Right-hand of it, as in the annexed Table. This being done, it will be easy to conceive, that the *Figures* in the small Column of the *Table*, do respectively represent those of the *Multiplier*: And that the *Numbers* against any of those *Figures* in the small Column, will be the true *Product* of the *Multiplicand* agreeing to any *Figure* of the *Multiplier*; as plainly appears by the Work of this *Example*.

1	879654
2	1759308
3	2638962
4	3518616
5	4398270
6	5277924
7	6157578
8	7037232
9	7916886
10	8796540

Then 879654, 79863 } The *Factors* as before.

Against	3, in the *Table* is	2638962	=879654×3
	6, is	5277924	=879654×60
	8, is	7037232	=879654×800
	9, is	7916886	=879654×9000
	7, is	6157578	=879654×70000
The Product required.		70251807402	=879654×79863

Note, This Method of Tabulating the *Multiplicand*, is both easy and certain; being neither subject to Errors, nor burdensome to the Memory, and therefore in large Calculations it may be found very useful. But for common Practice the usual Method (as in *Page* 18, &c.) is best, and to be preferred before this.

Most *Masters* that teach (and several Authors that write of) *Arithmetick*, do teach to prove the Truth of *Multiplication*, by casting away all the *Nines* that are contained in both the *Factors*, and their *Product*; but because that Method is very erronious, as might be easily shewed; I shall therefore omit inserting it, and leave the Proof of *Multiplication* to the next *Section*, wherein (I presume) the Reason and Proof, both of it, and *Division*, will plainly appear.

Sect.

Section 5. *Of* Divifion.

DIVISION is a *Rule* by which one *Number* may be fpeedily *fubftracted* from another, fo many Times as it is contained therein.

That is, it fpeedily difcovers how often one *Number* is contained (or may be found) in another: And to perform that, there are required two Numbers to be given.

1. The one of them is that *Number* which is propofed to be *divided*, and is called the *Dividend*.

2. The other is that *Number* by which the faid *Dividend* is to be *divided*, and is called the *Divifor*.

And by comparing thefe two, *viz.* the *Dividend* and the *Divifor* together, there will arife a third *Number*, called the *Quotient*; which fhews how often the *Divifor* is contained in the *Dividend*, or into what *Number* of equal Parts the *Dividend* is then divided. Therefore,

Divifion *is by* Euclid *fitly term'd the meafuring of one* Number *by another, viz.* one *Number* is faid to meafure another by that *Number*, which, when it *multiplies*, or is *multiplied* by it, it produceth. *Euclid* 7. *Def.* 23.

And if a Number *meafuring another*, multiply *that* Number *by which it meafureth, or be* multiplied *by it, it produceth the* Number *which it meafureth.* Euclid 7. Axiom 9.

That is to fay, if that *Number* which *divides* another (called the *Divifor*) be multiplied with the *Number* which is produced by *Divifion* (called the *Quotient*) their *Product* will be the *Number divided* or *Dividend*. Whence it follows, that *Divifion* and *Multiplication* are the converfe and direct contrary one to another (as *Subftraction* is to *Addition*) and do mutually prove the Truth of each other's *Operations*.

I fhall therefore make choice of the foregoing *Examples* in *Multiplication*, in order (as I prefume) to render the Bufinefs of *Divifion* more plain and eafy.

Firft, let it be required to find how often 6 is contained in 24. That is, to *divide* 24 by 6.

N. B. Always place down the given *Numbers* in this Order; Firft fet down the *Divifor*, and to the Right-hand of it draw a crooked Line; then fet down the *Dividend*, and to the Right-hand of it draw another crooked Line, in which muft be placed the *Quotient Figure*, or *Figures* as they become found.

Thus

Dividend.
Thus *Divisor* 6) 24 (4 the *Quotient.*

Here I consider how many Times 6 there is in 24, and find it 4, *viz.* 4 Times 6 is 24, therefore 4 is the true *Quotient* or *Answer* required.

This is apparent by *Substraction,* as in the Margin; where 24 the *Dividend* is set down, and from it 6 the *Divisor* continually *substracted* so often as it can be, which is just 4 Times. Therefore 4 is the true *Quotient* or *Answer* required.

Compare this with the *Example, page* 15.

	24
1	6
	18
2	6
	12
3	6
	6
4	6
	0

Corollary.

From hence it is evident, that *Division* is but a *concise* or *compendious Method* of *substracting* one *Number* from another, so often as it can be found therein; for if the *Divisor* be continually *substracted* from the *Dividend,* accounting an *Unit* (or 1) for each Time it is *substracted* (as above) the *Sum* of those *Units* will be the *Quotient.*

All Operations in *Division* do begin contrary to those of *Multiplication, viz.* at the first *Figure* to the Left-hand, or that of the highest Value, and decrease the *Dividend* by a repeated *Substraction* of each *Product* arising from the *Divisor* when *multiplied* into the *Quotient Figure.* And the only Difficulty in *Division* of whole *Numbers* (or indeed of any *Numbers)* lies in making choice of such a *Quotient Figure,* as is neither too big, nor too little; and that may be easily obtained by observing the following *Rule,* which hath two *Cases.*

RULE.

Case 1. *As often as the first* Figure *of the* Divisor *is taken from the first* Figure *of the* Dividend: *So often must the second* Figure *of the* Divisor *be taken from the second* Figure *of the* Dividend, *when it is joined with what* remains *of the first. And as often must the third* Figure *of the* Divisor *be taken from the third* Figure *of the* Dividend, *&c.*

But if the first *Figure* of the *Divisor* cannot be taken from the first *Figure* of the *Dividend*: Then,

Case 2. *So often as the first* Figure *of the* Divisor, *is taken from the two first* Figures *of the* Dividend, *so often must the second* Figure *of the* Divisor *be taken from the third* Figure *of the* Dividend, *when it is joined with what* remained *of the Second: And so often must the third* Figure *of the* Divisor *be taken from the fourth* Figure *of the* Dividend, *&c.*

That is, the *Quotient Figure* must be such, as being *multiplied* into the *Divisor*, will *produce* a *Product* equal to such a Part of the *Dividend* as is then taken for that *Operation:* But if such a *Product* cannot be exactly found, then the next less must be taken, and ordered, as in the following *Examples:* of which let that in *Page* 16 be the first, wherein there was given 8569 the *Multiplicand*, and 8 the *Multiplier*. To find the *Product* 68552. Let us here suppose the said *Product* 68552, and 8 the *Multiplier*, both given; thence to find the *Multiplicand*. That is, Let it be required to *divide* 68552 by 8.

Dividend.
Divisor 8) 68552 (*Quotient* when found.

According to the *Rule*, *Case* 1. I compare 8 the *Divisor* with 6 the first *Figure* of the *Dividend*, and finding I cannot take it from that; I then consider (by *Case* 2.) how often 8 can be taken from 68, the two first *Figures* of the *Dividend*, and find it may be taken 8 times; for 8 times 8 is 64, being the greatest *Product* of 8 (into any *Figure*) that can be taken from 68. I therefore place 8 in the *Quotient*, and with it *multiply* 8 the *Divisor*, setting down their *Product* underneath the said two first *Figures* of the *Dividend*, *substracting* it from them, and then the Work will stand

Thus 8) 68552 (8
64
———
4

In order to a second *Operation* I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 5, and bring it down underneath in its own Place to the *Remainder* 4, which will by that Means become 45. Then I consider how many Times 8 can be taken from 45, and find it may be 5 times; for 5 times 8 is 40, I therefore place 5 in the *Quotient*, and with it *multiply* 8 the *Divisor*, setting down and *substracting* their *Product*, as before. Then the Work will stand

Thus

```
Thus 8) 68552 (85
        64.
        ---
        45
        40
        ---
         5
```

For a third *Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 5, and bring it down, as before, proceeding in all refpects, as before; and then the Work will ftand

```
Thus 8) 68552 (856
        64..
        ---
        45
        40
        ---
         55
         48
         ---
          7
```

Laftly, I point and bring down the 2, *viz.* the laft *Figure* of the *Dividend* to the *Remainder* 7, which will then become 72, and proceeding as in the other *Operations*, I find that 8 the *Divifor*, can be taken juft 9 Times from 72, and the Work is finifhed, and will ftand

```
Thus 8) 68552 (8569
        64...
        ---
        45
        40
        ---
         55
         48
         ---
          72
          72
         ---
         (0)
```

The true *Quotient* is found to be 8569, being exactly the *eighth* Part of 68552, or the *Multiplicand* of the propofed *Example* of *Multiplication*. As was required.

The Reafon of the *Operations* will be very plain to any one that will a little confider it, as follows,

Divifor

Divisor 8) 68552 (8000 The first *Quotient Figure.*

Substract	64000	This *Product* of the *Divisor* into the *Quotient* is 64000, *viz.* 8 times 8000; the *Quotient Figure* being always of the same *Value*, or *Degree*, with that *Figure*, under which the *Unit*'s place of its *Product* stands.
Divisor 8)	4552	(500. The second *Quotient Figure.*
Substract	4000	And here the *Product* is 4000, *viz.* 8 times 500, not 8 times 5.
Divisor 8)	552	(60. The third *Quotient Figure.*
Substract	480	Also here the *Product* is 480, *viz.* 8 times 60, for the *Reasons* abovesaid.
Divisor 8)	72	(9. The fourth *Quotient Figure.*
Substract	72	Now here the *Product* is but 72, *viz.* 9 times 8, because the 9 stands in the Place of *Units.*
Remains	(0 0)	Now the *Sum* of all the several *Quotients,*

viz. 8000+500+60+9=8569, as before.

If the *Process* of this *Example* be well considered and compared with that of *Multiplication, Page* 17, it will evidently appear to be only the *Converse* of that; for the particular *Products* are alike in both, only that which is *last* there, is *first* here; there they are *added*; here they are *substracted.* So that whoever understands the true *Reason* of the one, must needs understand the *Reason* of the other, and then *Division* will become very *easy,* although the *Divisor* consists of several Places of *Figures.*

EXAMPLE.

Let it be required to *divide* 590624922 by 7563.

Dividend

Divisor 7563) 590624922 (

'Tis plain at the first sight, that 7563 the *Divisor,* cannot be taken from 5906, the like *Number* of *Figures* in the *Dividend.*

Therefore, by the *Second Case* of the *Rule (Page* 23.) there must be allowed five *Figures* of the *Dividend, viz.* 59062 for the *first Operation* or *Quotient*; that so the *first Figure* 7 of the *Divisor* may be taken out of the two *first Figures,* viz. 59 of the *Dividend,* &c.

Then I proceed (*per Case* 2.) and consider how often 7 may be taken from 59, and find it may be taken 8 times, for 8 times 7 is but 56, which I mentally *substract* from 59, and there *remains* 3; to this three I mentally adjoin the *third Figure* of the *Dividend*, viz. 0, which makes it 30, out of which I must take the *second Figure* of the *Divisor*, viz. 5, so often as I took the 7 from 59, which was 8 times: But that cannot be, for 8 times 5 is 40, which is more than 30, therefore 8 is too big a *Figure* to be placed in the *Quotient*; yet, hence I conclude, that the next less, *viz.* 7 may be taken without any further *Trial*. I therefore place 7 in the *Quotient*, and with it *multiply* the *Divisor*, setting down their *Product* under the *Dividend*, and *substract* it from thence, as in the other *Example*, and then the *Work* will stand.

```
Thus 7563) 590624922 (7
           52941
           -----
            6121
```

In order to a *second Operation*, I make a *Point* under the next *Figure* of the *Dividend*, viz. under the 4, and bring it down to the *Remainder* 6121, which will then become 61214, with which I proceed in all respects as I did before with the 59062, and find the next *Quotient Figure* will be 8, with which I *multiply* the *Divisor*, &c. and *substract* their *Product* from the said 61214. Then the *Work* will stand

```
Thus 7563) 590624922 (78
           52941 .
           ------
            61214
            60504
           ------
              710
```

To this *Remainder* 710, I point and bring down the next *Figure* of the *Dividend*, viz. 9, which makes it 7109; now because the *Divisor* 7563 cannot be taken from 7109, I therefore place a *Cypher* in the *Quotient*.

And this must always be carefully observed, viz. That for every Figure *or* Cypher, *which is brought down from the* Dividend, *in order to a new* Operation, *there must always be either a* Figure *or* Cypher, *set down in the* Quotient. Then the Work will stand

Thus

Thus 7563) 590624922 (780
52941..

61214
60504

7109

To this 7109, I bring down another *Figure* of the *Dividend*, viz. 2. and then it will become 71092; then I confider how often 7 can be taken from 71, &c. (juft as at the firft Operation) and find it may be taken 9 times, therefore I fet down 9 in the *Quotient*, and with it *multiply* the *Divifor*, fetting down and *fubftracting* their *Product*, as before; then the Work will ftand

Thus 7563) 590624922 (7809
52941...

61214
60504

71092
68067

3025

To this *Remainder* 3025, I point and bring down the laft *Figure* 2 of the *Dividend*, which makes it 30252; then proceeding in all refpects as before, I find the *Quotient Figure* to be 4, with it I *multiply* the *Divifor*, fetting down and *fubftracting* their *Product* as before, and then the Work will ftand

Thus 7563) 590624922 (78094
52941....

61214
60504

71092
68067

30252
30252

(00000)

Here the Work is ended, and I find the *Quotient* to be 78094, being the true *Multiplicand* of the propos'd *Example* of *Multiplication*, Page 18.

That is, 7563 is contained in 590624922 just 78094 times, *&c.*

If the Work of this *Example* be considered and compared with the *Rule (Page 22.)* the whole Business of *Division* will be easy; for indeed the only Difficulty, as I said before, lies in making choice of a true *Quotient Figure*, which cannot well be done according to the common Method of *Division* without Trials, yet those Trials need not be made with the whole *Divisor*, (as appears by this last *Example)* for by the two first *Figures* of the *Divisor* all the rest are generally regulated; except the second *Figure* chance to be 2, 3, or 4, and at the same time the third *Figure* be 7, 8, or 9, then indeed Respect must be had to the third *Figure*, according as the *Rule* directs.

However, if those Trials are thought too troublesome, they may be avoided, and the same *Quotient Figure* may both easily and certainly be found by help of such a small *Table* made of the *Divisor*, as was of the *Multiplicand* in *Page* 20.

EXAMPLE 4.

Let it be required to *divide* 70251807402 by 79863. *See the Example of Multiplication, Page 20, and as there directed make a Table of the* Divisor 79863,

Thus

	Divisor.	*Dividend.*	*Quotient.*
1	79863)	70251807402	(879654
2	159726	638904.....	
3	239589	636140	
4	319452	559041	
5	399315	770997	
6	479178	718767	
7	559041	522304	
8	638904	479178	
9	718767	431260	
10	798630	399315	
		319452	
		319452	
		(000000)	

The Work of this Operation I presume may be easily understood. For those *Figures* in the *Table* are the *Product* of the *Divisor* into all the 9 *Figures*; consequently those *Figures* in the small Column do shew what *Figure* is to be placed in the *Quotient*, without any doubtful Trials of the *Divisor*, with the *Dividend*, as before.

This Method of Tabulating the *Divisor* may be of good Use to a Learner; especially until he is well practised in *Division*; yea, and even then if the *Divisor* be large, and a *Quotient* of many *Figures* be required, as in resolving of high *Equations*, and calculating of *Astronomical Tables*, or those of Interest, *&c.*

Hitherto

Hitherto I have made choice of *Examples* wherein the *Dividend* is truly meafured or *divided* off by the *Divifor*, without leaving any *Remainder*, being exactly compofed of the *Divifor* and *Quotient*. But it moft ufually falls out, that the *Divifor* will not exactly meafure the *Dividend*; in which Cafe the *Remainder* (after *Divifion* is ended) muft be fet over the *Divifor* with a fmall Line betwixt them adjoining to the *Quotient*.

EXAMPLE 5.

Suppofe it were required to *divide* 379 by 5.

$$
\begin{array}{rl}
5)\ 379\ (75\frac{4}{5} & \begin{array}{l}\text{the Remainder}\\ \text{the Divifor.}\end{array}\\
35\cdot\ \ \ \ \ \ & \\
\hline
29\ \ \ \ \ \ & \\
25\ \ \ \ \ \ & \\
\hline
\textit{Remains}\ (4)\ \ \ \ \ \ &
\end{array}
$$

EXAMPLE 6.

Again, Let it be required to *divide* 43789 by 67.

$$
\begin{array}{rl}
67)\ 43789\ (653\frac{38}{67} & \text{the true \textit{Quotient} required.}\\
402\cdot\cdot\ \ \ \ \ \ \ & \\
\hline
358\ \ \ \ \ \ \ & \\
335\ \ \ \ \ \ \ & \\
\hline
239\ \ \ \ \ \ \ & \\
201\ \ \ \ \ \ \ & \\
\hline
\textit{Remains}\ (38)\ \ \ \ \ \ \ &
\end{array}
$$

How fuch *Remainders* thus placed over their *Divifors* (which are indeed *Vulgar Fractions*) may be otherwife managed, fhall be fhewed farther on.

N. B. When the *Divifor* happens to be an *Unit*, viz. 1, with a *Cypher* or *Cyphers* annexed to it, as 10, 100, 1000, &c. *Divifion* is truly performed by cutting off with a Point or Comma, fo many *Figures* of the *Dividend* as there are *Cyphers* in the *Divifor*; then are thofe *Figures* fo cut off to be accounted a *Remainder*, and the reft of the *Figures* in the *Dividend* will be the true Quotient required, becaufe an *Unit* or 1 doth neither *multiply* nor *divide*.

EXAMPLE 7.

Let it be required to *divide* 57842 by 100. The Work may ftand thus, 100) 578,42 the *Quotient* required; or thus, 100) 57842 ($578\frac{42}{100}$ the fame as before.

Hence

Hence it follows, that if any *Divisor* have *Cyphers* to the Right hand of it, you may cut off so many of the last *Figures* in the *Dividend*, and *divide* the other *Figures* of the *Dividend*, by those *Figures* of the *Divisor* that are left when the *Cyphers* are omitted. But when *Division* is ended, those *Cyphers* so omitted in the *Divisor*, and the *Figures* cut off in the *Dividend*, are both to be restored to their own Places.

EXAMPLE 8.

Suppose it were required to *divide* 675469 by 5400.

```
5400) 675469 (125
 ——     ——
      54··
      ————
      135
      108
      ————
       274
       270
      ————
```

Remains (4) But the true *Remainder* is 469.

Consequently the true *Quotient* is $125\frac{469}{5400}$.

As to the Manner of proving the Truth of any Operation, either in *Multiplication* or *Division*, I presume it may be easily understood, by what is delivered in *Page* 21, compared with the three first *Examples* of *Division*; for from thence it will be easy to conceive, that if the *Divisor* and *Quotient* be *multiplied* together, their *Product* (with what *remains* after *Division* being added to that *Product*) will be equal to the *Dividend*. As in the fifth *Example*, where the *Dividend* is 379, the *Divisor* is 5, the *Quotient* is 75, and the *Remainder* is 4.

I say, $75\times5=375$, to which add the *Remainder* 4, it will be 379.

Again, in the sixth *Example*, the *Divisor* is 67, the *Quotient* is 653, and the *Remainder* is 38.

Then $653\times67=43751$, and $43751+38=43789$ the *Dividend*, &c.

There are several useful Contractions both in Division *and* Multiplication, *which I have purposely omitted until I come to treat of* Decimal Arithmetick. Also I have omitted the Business of Evolution or extracting of *Roots*, until further on; and so shall conclude this *Chapter* with a few *Examples* of *Division* unwrought at large, leaving them for the Learner's Practice.

579)

579) 43800771 (75649.
Or 75649) 43800771 (579.
45007) 23884044718 (530674.
Or 530674) 23884044718 (45007.
356) 244572000 (687000..
59600) 57659066400 (967434.
10000) 679543820000 (67954382.
79) 282016 (3569 $\frac{45}{79}$.

CHAP. III.

Concerning ADDITION *and* SUBSTRACTION *of* Numbers *of different* Denominations, *and how to* reduce *them from one* Denomination *to another*.

SECT. I.

1. *Of* English *Coin*.

THE least Piece of *Money* used in *England* is a *Farthing*, and from thence ariseth the rest, as in this *Table*.

Farth.
4= 1*d. Penny.*
48= 12= 1*s. Shilling.*
960=240=20 1*l. Pound Sterling.*

And { 5*s.* is a *Crown.* 10*s.* is an *Angel.* 6*s.* 8*d.* a *Noble.* 13*s.* 4*d.* a *Mark.* }

Note, When *l. s. d. q.* are placed over (*or to the Right-hand of*) *Numbers*, they denote those *Numbers* to signify *Pounds*, *Shillings*, *Pence*, and *Farthings*.

l. *s.* *d.* *q.*
As 35 10 6. 2. Or 35*l.* 10*s.* 6¼. Either of these do signify 35 *Pounds*, 10 *Shillings*, 6 *Pence*, 2 *Farthings*.

The same must be understood of all the following *Characters* belonging to their respective *Tables*, viz. Of *Weights*, *Measures*, &c.

2. *Troy Weight.*

The Original of all *Weights* used in *England*, was a *Corn* of *Wheat* gathered out of the *Middle* of the *Ear*, and being well dried, 32 of them were to make one *Penny Weight*, 20 *Penny Weight* one *Ounce*, and 12 *Ounces* one *Pound Troy*. *Vide Statutes* of 51 *Hen.* III. 31 *Edw.* I. 12 *Hen.* VII.

But

But in later Times it was thought ſufficient to *divide* the aforeſaid *Penny Weight* into 24 equal *Parts*, called *Grains*, being the leaſt *Weight* now in common Uſe; and from thence the reſt is computed as in this *Table*.

Gr. Grain.			
24=	1 P. W. Penny Weight.		
480=	20=	1 Oz. Ounce.	
5760=	240=	12=	1 ℔ Pound.

Note, By *Troy Weight* are weighed *Jewels*, *Gold*, *Silver*, *Corn*, *Bread*, and all *Liquors*.

Beſides the common *Diviſions* of *Troy Weight*, I find in *Angliæ Notitia*, or, *The preſent State of England*, printed in the Year 1699, that the *Moneyers (as that Author calls them)* do *ſubdivide* the *Grain*.

Thus:
- 24 *Blanks*=1 *Periot*.
- 20 *Periots*=1 *Droite*.
- 24 *Droits*=1 *Mite*.
- 20 *Mites*=1 *Grain*, &c. as before.

3. *Apothecaries Weights*.

The *Apothecaries divide* a *Pound Troy* as in this *Table*.

Gr. Grain.				
20=	1 ℈ Scruple			
60=	3=	1 ʒ Dram		
480=	24=	8=	1 ℥ Ounce	
5760=	288=	96=	12	1 ℔ *Troy*, the ſame as before.

By theſe *Weights* the *Apothecaries* compound their Medicines: but buy and ſell their Drugs by *Averdupois Weight*.

4. *Averdupois Weight*.

When *Averdupois Weight* became firſt in *Uſe*, or by what *Law* it was firſt ſettled, I cannot find out in the *Statute Books*; but on the contrary, I find that there ſhould be but one *Weight (and one Meaſure)* uſed throughout this *Realm*, viz. that of *Troy*, *(Vide* 14 *Ed*. III. and 17 *Ed*. III.) So that it ſeems *(to me)* to be firſt introduced by *Chance*, and ſettled by *Cuſtom*, viz. from giving good or large *Weight* to thoſe *Commodities* uſually weighed by it, which are ſuch as are either very *coarſe* and *droſſy*, or

very

very subject to waste; as all Kind of Grocery Wares. And *Pitch, Tar, Rosin, Wax, Tallow, Flax, Hemp,* &c. *Copper, Tin, Steel, Iron, Lead,* &c. Also *Flesh, Butter, Cheese, Salt,* &c. To these and the like, I presume, it was thought convenient to allow a greater *Weight* than the Laws had provided, which happened to be about a sixth Part more: For I found by a very nice Experiment, that one *Pound Averdupois* is equal to 14 *Ounces,* 11 *Penny Weight,* and $15\frac{1}{2}$ *Grains Troy.* And it is now computed as in the following Table.

Drams.				
16=	1 *Oz. Ounces.*			
256=	16=	1 ℔ *Pounds.*		
28672=	1792=	112=	1 *C. Hundred.*	
573440=	35840=	2240=	20=	1 *Tun.*

	℔
And	14=*a Stone.*
	28=$\frac{1}{4}$ *of C.*
	56=$\frac{1}{2}$ *of C.*
	84=$\frac{3}{4}$ *of C.*

5. *Long Measure.*

As the least Part of *Weight* came at first from a *Wheat Corn,* so (it is generally said) the least Part of a *Long Measure* was at first a *Barley Corn,* taken out of the Middle of the Ear, and being well dried, three of them in Length were to make one *Inch*; and thence the rest, as in this Table.

Barley Corns.						
3=	1 *In. Inches.*					
36=	12=	1 *F. Feet.*				
108=	36=	3=	1 *Y. Yards.*			
594=	198=	$16\frac{1}{2}$=	$5\frac{1}{2}$=	1 *P. Poles.*		
23760=	7920=	660=	220=	40=	1 *Furlong.*	
190080=	63360=	5280=	1760=	320=	8=	1 *Mile.*

And	4 *Nails*=$\frac{1}{4}$ of a *Yard.*
	$1\frac{1}{4}$ *Yard*= 1 *Ell.*
	2 *Yards*=1 *Fathom.*

Note, That forty Statute *Poles,* or *Perches,* in Length, and four in Breadth, do make a Statute *Acre* of *Land.*

That is, 220 *Yards, multiplied* into 22 *Yards*=4840 square *Yards* are a Statute *Acre.*

And according to the Transactions of the *French Academy, Anno* 1687, a *Paris Foot Royal* is=12,8 *Inches English*; Six of those Feet make a *Toise*; and 57060 *Toises*=365184 *English Feet,* are the *Measure* of one Degree of a great Circle upon the Surface of the Earth. So that one Degree is 69 *Miles* and 288 *Yards,* which is very near to our Country-man Mr. *Norwood's* Experiment made betwixt *London* and *York, Anno* 1635; who found that 367196 *Feet*=69 *Miles,* and 958 *Yards* do make a

Degree. And not 60 *Miles*, according to the common received Opinion and Practice of the *Navigators* or *Seamen*.

Hence, according to the *French* Account, the Circumference of the Earth (supposing it to be a true *spherical Figure*) is 24899 *English Miles*.

6. Of *Liquid Measures*.

All *Measures* of Capacity, both Liquid and Dry, were at first made from *Troy Weight*, *Vide Statutes* 9 *H.* III. 51 *H.* III. 12 *H.* VII. *&c.* wherein it is enacted, that eight *Pound Troy Weight*, of *Wheat*, gathered out of the Middle of the Ear, and well dried, should make one *Gallon* of *Wine Measure:* And that there should be but one *Measure* for *Wine*, *Ale*, and *Corn*, throughout this Realm, (*Vid. Stat.* 14 *Edw.* III. 15 *Rich.* II.) But Time and Custom have altered *Measures*, as they have done *Weights* (and perhaps for one and the same Reason) for now we have three different *Measures*, *viz.* one for *Wine*, one for *Ale* or *Beer*, and one for *Corn*.

I have inserted *Tables* of each, as they are now computed by *Cubick Inches*, and practised in the Art of *Gauging*, &c.

The common *Wine Gallon* sealed at *Guild-hall* in *London*; by which all *Wines*, *Brandies*, *Spirits*, *Strong-waters*, *Mead*, *Perry*, *Cyder*, *Vinegar*, *Oil*, and *Honey*, &c. are measured and sold; is supposed to contain 231 *Cubick Inches*, and from thence the rest are computed, as in this Table.

Cubick Inches.

231=1 *G. Gallons.*
9702= 42=1 *Terce.*
14553= 63=$1\frac{1}{2}$=1 *Hogshead.*
19404= 84=2 =$1\frac{1}{3}$=1 *Puncion.*
29106=126=3 =2 =$1\frac{1}{2}$=1 *Butt or Pipe.*
58212=252=6 =4 =3 =2=1 *Tun.*

Gallons.

Note, $\left\{\begin{array}{l}\text{18=1 } \textit{Runlet}\text{, and}\\ 31\frac{1}{2}\text{ makes a } \textit{Wine}\\ \text{or } \textit{Vinegar Barrel.}\\ (\textit{Vide}\text{ 1 } \textit{R.}\text{ III.}\end{array}\right.$

But Dr. *Wybard* in his Tectometry, *page* 289, doth suppose the *Wine Gallon* to contain but 224, or 225 *Cubick Inches* at the most, and pursuant to this Account an Experiment was made by Mr. *Richard Walker*, and Mr. *Philip Shales*, two General Officers in the Excise. They caused a Vessel to be very exactly made of Brass in Form of a *Parallopipedon*, each Side of its Base was 4 *Inches*, and its Depth 14 *Inches*; so that it's just Content was 224 *Cubick Inches*. This Vessel was produced at *Guild-Hall* in *London* (*May* 25. 1688.) before the *Lord-Mayor*, the *Commissioners* of *Excise*, the Revd. Mr. *Flamstead*, Astr. Reg. Mr.

Mr. *Halley*, and feveral other ingenious Gentlemen, in whofe Prefence Mr. *Shales* did exactly fill the aforefaid brazen Veffel with clear Water, and very carefully emptied it into the old Standard *Wine Gallon* kept in *Guild-hall*, which did fo exactly fill it, that all then prefent were fully fatisfied the *Wine Gallon* doth contain but 224 *Cubick Inches*. (*This notable Experiment I faw tried.*) However, for feveral Reafons, it was at that Time thought convenient to continue the former fuppofed Content of 231 *Cubick Inches* to be the *Wine Gallon*, and that all Computations in *Gauging* fhould be made from thence, as above.

The *Beer* or *Ale Gallon* (which are both one) is much larger than the *Wine Gallon*; it being (as I prefume) made at firft to correfpond with *Averdupois Weight*, as the *Wine Gallon* did with *Troy Weight*: For (as I faid before, *page* 33.) one *Pound Averdupois* is equal to 14 *Ounces* 12 *Penny Weight Troy*, very near.

And, as one *Pound Troy* is in proportion to the *Cubick Inches* in a *Wine Gallon*, fo is one *Pound Averdupois* to the *Cubick Inches* in an *Ale Gallon*. That is, $12 : 231 :: 14\frac{12}{20} : 281\frac{1}{8}$, very near the *Cubick Inches* contained in an *Ale Gallon*, as appears from an Experiment made by one *Nicholas Gunton*, General *Gauger* in the *Excife*, about 41 Years ago, who, by fuch a Veffel mentioned before in the laft *Page*, did find the Standard *Ale Quart* (kept in the *Exchequer*, *Vid.* 12 *Car.* II.) to contain juft $70\frac{1}{2}$ *Cubick Inches*, confequently the *Ale Gallon* muft contain 282 *Cubick Inches*, and from thence the following *Tables* are computed.

Ale Meafure.

Cubick Inches.

282 = 1 *Gallon.*
2256 = 8 = 1 *Firkin.*
4512 = 16 = 2 = 1 *Kilderkin.*
9024 = 32 = 4 = 2 = 1 *Barrel.*
13536 = 48 = 6 = 3 = 1 = $\frac{1}{2}$1 *Hogfhead.*

Note, { *A Firkin of Soap and of Herrings are the fame with that of Ale.* }

Beer Meafure.

Cubick Inches.

282 = 1 *Gallon.*
2538 = 9 = 1 *Firkin.*
5076 = 18 = 2 = 1 *Kilderkin.*
10152 = 36 = 4 = 2 = 1 *Barrel.*
15228 = 54 = 6 = 3 = $1\frac{1}{2}$ = 1 *Hogfhead.*

N. B. This Diſtinction or Difference betwixt *Ale* and *Beer Meaſure*, is now only uſed in *London*. But in all other Places of *England*, the following *Table* of *Beer* or *Ale*, whether it be ſtrong or ſmall, is to be obſerved, according to a Statute of *Exciſe* made in the Year 1689.

Cubick Inches.

282=1 *Gallon.*

2397=8½=1 *Firkin.*

4794=17=2=1 *Kilderkin.*

9588=34=4=2=1 *Barrel.*

14382=51=6=3=1½=1 *Hogſhead.*

7. Of *Dry Meaſure.*

Dry Meaſure is different both from *Wine* and *Ale Meaſure*, being as it were a Mean betwixt both, tho' not exactly ſo; which upon Examination I find to be in Proportion to the aforeſaid old Standard *Wine Gallon*, as *Averdupois Weight* is to *Troy Weight*; That is, As one *Pound Troy* is to one *Pound Averdupois*, ſo is the *Cubick Inches* contained in the old *Wine Gallon*: To the *Cubick Inches* contained in the *Dry* or *Corn Gallon*.

Viz. 12 : $14\frac{12}{20}$: : 224 : $272\frac{1}{2}$, which is very near to $272\frac{1}{4}$, the common received Content of a *Corn Gallon*: Altho' now it is otherwiſe ſettled by an Act of Parliament made in *April* 1697, the Words of that Act are theſe:

Every round Buſhel *with a plain and even Bottom, being made eighteen* Inches *and a half wide throughout, and eight* Inches *deep, ſhould be eſteemed a legal* Wincheſter Buſhel, *according to the Standard in his* Majeſty's Exchequer.

Now a Veſſel being thus made will contain 2150,42 *Cubick Inches*, conſequently the *Corn Gallon* doth contain but $268\frac{4}{5}$ *Cubick Inches.*

Cubick Inches.

268,8= 1 *Gallon.*

537,6= 2= 1 *Peck.*

2150,4= 8= 4=1 *Buſhel.*

17203,2=64=32=8=1 *Quarter.*

Note, { 4 *Buſhels*=a *Comb.*
10 *Quarters*=a *Wey*, and
12 *Weys*=a *Laſt of Corn.* }

I obſerved amongſt the Lead-Mines in *Derbyſhire*, (*Anno* 1692) that the *Miners* bought and ſold their Lead Ore, by a *Meaſure* which they called an Ore Diſh; whoſe Dimenſions I carefully took, and found it

Thus { Length 21.3.
Breadth 6.
Depth 8.4. } *Inches.*

Conſe-

Confequently its Content is 1073,52 *Cubick Inches*, which is very near equal to 4 *Corn Gallons*, according to the abovementioned Settlement.

Nine of thofe Difhes they call a Load of Ore, which if it be pretty good, will produce about 3 hundred Weight of Lead.

8. Of *Time*.

It is not an eafy Thing to give a true *Definition* of *Time*; for (according to the *philofophick Poet.)*

> Time *of itfelf is nothing, but from Thought*
> *Receives its Rife, by labouring Fancy wrought*
> *From Things confider'd, whilft we think on fome*
> *As prefent, fome as paft, or yet to come.*
> *No Thought can think on* Time, *that's ftill confeft,*
> *But thinks on Things in Motion or at Reft.*

And fo on, Vide *Lucretius*, Book I.

That is, *Time* only fhews the *Duration* or *Mutation* of Things, a Year being the *Standard* or *Integer*, by which fuch Continuance or Change is computed. And a *Year* is that *Space* of *Time* in which the *Sun* (apparently) compleats its *Revolution* from any one *Point* in the *Ecliptick* (an imaginary *Circle* in the *Heavens)* to the fame *Point* again, which, according to modern *Obfervations*, is performed in 365 *Days*, 5 *Hours*, 48 *Minutes*, 57 *Seconds*. 21 *Thirds*, &c. But a *Second* being the leaft Part of *Time* that can be truly *meafured* by the *Motion* of any *mechanical Engine*, as a *Clock*, &c. (a *Third* being lefs than the Twinkling of an Eye) I begin the following Table with *Seconds*.

Seconds.″				
60=1′	*Minute.*			
3600=	60=1°	*Hour.*		
86400=	1440=	24=1	*Day.*	
31556937=	525948=	8765=	$365^{\circ}+5+48'+57''$=1	*Year, called a Solar Year.*

But the common *Year*, ufually called the *Julian Year*, doth confift of 365 *Days* and 6 *Hours*, and is divided into twelve unequal *Months*, called *Kalendar Months*, whofe *Names* and *Number* of *Days* are the Subject of every *Almanack*.

To

To these *Tables* it may not be amiss to give a brief Account of such *Coins*, *Weights*, and *Measures*, as are frequently mentioned in the Scriptures. As I have deduced them from those which seem to be the most correct, inserted in the *Index* to the large *Bible*, printed *Anno* 1702, and compared with those used in *England*, by the Lord Bishop of *Peterborough*.

The *Hebrew Weights*, compared with { *Troy Weight.*

	Oz.	Pw.	Grains.
A Gerah=	0	0	10 1/8
10 *Gerahs*=*a Bekah*=	0	4	13½
2 *Bekahs*=*a Shekel*=	0	9	3
100 *Shekels*=*a Menah*=	45	12	12

Note, A *Shekel* is said to be their original *Weight*.

Their *Coin* { *English Coin.*

	l.	s.	d.	
A *Silver Menah*=	7	1	5	Weight 60 *Shekels*.
Talent of *Silver*=	357	11	10½	Weight 3000 *Shekels*.
Talent of *Gold*=	5075	15	7½	The same Weight men-
The *Gold Dram*=	1	0	4	tioned *Ex.* xxv. 39.

The *Roman Money* mentioned in the *New Testament*.

A *Denarius*, or *Silver Penny*=7*d*. 3 *Farthings*.
Asses of *Copper*=0 . 3 *Farthings*.
Assarium=0 . 1½ *Farthing*.
Quadrans=0 . ¾ of a *Farthing*.
A Mite=0 . ⅓ of a *Farthing*.

Their *Long Measure* compared with { *English Measure.*

	Yar.	Feet.	In. Pts.
A Finger's Breadth=	0.	0	0,912
4 *Fingers*=*a Hand's Breadth*=	0.	0	3,648
2 *Hands*=*the least Span*=	0.	0	7,296
3 *Hands Breadth*=*the longest Span*=	0.	0	10,944
2 *Spans*=*the longest Cubit*=	0.	1	9,888
4 *Cubits*=*a Fathom*=	2.	1	3,552
6 *Cubits*=*Ezekiel's Reed*=	3.	1	11,328
400 *Cubits*=*a Stadium*=	243.	0	7,2
10 *Stadiums*=*a Mile*=	2432.	0	0
3 *Miles*=*a Parasang*=	7296.	0	0
Which is 4 *English Miles* and	256.		

Their

Their *Measures* of *Capacity*, compared with	English Wine. Gal.	Pints.	Inch.
A Cotyla=	0	. $0\frac{1}{2}$	3,037
A Log=	0	. $0\frac{1}{4}$	9,83
4 *Logs=a Cab*=	0	. 3 .	10,458
10 *Cotyla's=an Omer*=	0	. 6 .	1,5
3 *Cabs=a Hin*=	1	. 2 .	2,5
2 *Hins=a Seab*=	2	. 4 .	5,
3 *Seabs=an Epha*=	7	. 4 .	15,
10 *Epha's=a Chomer*=	75	. 5 .	5,625

Sect. 2. Addition of *Weights*, &c.

The foregoing *Tables* being so well understood, as that you can really tell, without pausing, how many *Units* of any one *Denomination*, do make one of the next *superior Denomination (especially in those* Tables *which are most useful for your Business)* it will then be as easy to *add* or *substract* them, as to *add* or *substract whole Numbers*, due Care being taken in placing all *Numbers* that are of one *Denomination* exactly underneath each other. That is to say, in *Money*, place *Pounds* under *Pounds*, *Shillings* under *Shillings*, *Pence* under *Pence*, &c. Understand the like in *Weights* and *Measures*, &c. according to their several *Denominations:* Then in *Addition* observe this *Rule*.

RULE.

Alaways begin with those Figures *of the lowest or least* Denomination, *and* add *them all together into one* Sum, *then consider how many of the next* superior Denomination *are contained in that* Sum, *so many Units you must carry to the said next* superior Denomination *to be added together with those* Figures *that stand there*; *and if any thing remain over or above those* Units *so carried, that* Overplus *must be set down underneath its own* Denomination: *And so proceed on from one* Denomination *to another until all be finished.*

Example in Coin.

Let it be required to *add* 35*l.* 14*s.* 06*d.* and 27*l.* 02*s.* 10*d.* and 54*l.* 13*s.* 04*d.* and 10*l.* 17*s.* 09*d.* into one *Sum.*

The particular *Sums* being placed, as before directed, will stand as in the Margin following.

Then according to the *Rule*, I begin with the *Pence* (being here the lowest or least *Denomination)* and *adding* them all together, I find their *Sum* to be 29*d.* that is 2*s.* and 5*d.* (for 24=2*s.*

and

and 29—24=5) the 5*d*. I set down underneath its own *Denomination*, and carry the 2*s*. to the Place of *Shillings*, *adding* them and all the *Shillings* together, I find the *Sum* to be 48*s*. *viz*. 2*l*. 8*s*. I set down the 8*s*. underneath its own Place of *Shillings*, and carry the 2*l*. to the Place of *Pounds*, *adding* them and all the *Pounds* together, I find their *Sum* is 128*l*. consequently the *Total Sum* required is 128*l*. 08*s*. 05*d*.

l.	*s.*	*d.*
35	14	06
27	02	10
54	13	04
10	17	09
128	08	05

Now, for as much as it often happens in keeping Books of *Accounts*, (and in other Business) that it is required to *add* up large *Sums* of Money, consisting of 30, 40, or more several particular *Sums*, nay, perhaps filling up the whole Length of a Sheet of Paper, I humbly conceive in those Cases the best and easiest Way will be to part them into Parcels, not exceeding above 10 or 12 particular *Sums* in each Parcel; that done, add together all the *Sums* of those Parcels into one *Sum*, and that will be the *Total Sum* required.

Also to avoid the making of *Points*, or other *Marks* amongst your *Figures*, it will be convenient to get the following *Tables* by heart.

The Pence Table.		*The* Shillings Table.	
d. *s.*	*d.* *s.*	*s.* *l.*	*s.* *l.*
12=1	72= 6	20=1	120= 6
24=2	84= 7	40=2	140= 7
36=3	96= 8	60=3	160= 8
48=4	108= 9	80=4	180= 9
60=5	120=10	100=5	200=10

The Use of these *Tables* is so obvious, that I presume it is needless to explain them.

Examples in Addition *of* Weights.

Troy Weight.

	lb.	*Oz.*	*Pw.*	*Gr.*
	3	09	00	10
	5	08	15	21
	10	10	12	22
	0	11	19	23
Sum	21	04	09	04

Averdupois Weight.

	Tun.	*C.*	*Q.*	lb.	*Oz.*
	12	15	2	24	12
	7	10	3	21	15
	0	18	1	14	11
	1	19	3	27	15
Sum	23	05	0	05	05

Examples

Examples in Addition of long Measure.

Yards.	*Qrs.*	*Nails.*		*Miles.*	*Fur.*	*Poles.*	*Yards.*	*Feet.*	*Inch.*
35	2	3		2	6	32	4	2	9
17	3	1		0	7	27	3	1	10
129	1	2		1	3	39	$1\frac{1}{2}$	2	11
182	3	2	Sum	5	2	19	5	1	6

I think it needless to set down more Examples of this Kind, for if these 5, especially the last, will be understood, they will be sufficient to shew how any other may be performed.

Sect. 3. SUBSTRACTION *of Weights*, &c.

SUbstraction is but the Converse of the precedent Work, and may be performed by observing this *Rule*.

RULE.

Begin with the lowest or least Denomination, as before in Addition, and take or substract the Figure, or Figures, in that Place of the Subtrahend, from the Figure or Figures, that stand over them of the same Denomination; setting down the Remainder (as in *page* 12.) But if that cannot be done, then you must increase the upper Figure or Figures, with one of the next superior Denomination, and from that Sum make Substraction; and so proceed to the next superior Denomination, where you must pay the one borrowed, by adding Unity to the Subtrahend in that Place, *&c.* as in whole Numbers.

Example in Coin.

	l.	*s.*	*d.*		*l.*	*s.*	*d.*
From	386	09	08	From	569	10	06
Take	173	04	06	Subt.	389	15	08
Remains	213	05	02		179	14	10

The first of these Examples is self-evident. In the second Example, beginning at the Place of Pence, being here the least Denomination, I am to take 8*d.* from 6*d.* but because that cannot be done, I must, according to the Rule, borrow one of the next Denomination, *viz.* 1*s.* and add it to the 6*d.* which makes it 18*d.* (for 1*s.*=12*d.* and 12+6*d.*=18*d.*) then I take 8*d.* from that 18*d.* and there remains 10*d.* to be set down underneath the Place of Pence; that done, I proceed to the Place of Shillings, where I must now pay the 1*s.* saying one borrowed and 15 makes 16 from 10 cannot be, but 16 from 30 and there remains 14. That is, I

borrow

borrow one of the next Denomination, *viz.* 1*l.* and add to it the 10*s.* which makes it 30*s.* (for 1*l.*=20*s.* and 20+10=30) having set down the remaining 14*s.* underneath its own Place of Shillings, I proceed to the Place of Pounds, where paying the 1*l.* borrowed, it will be 1 borrowed and 9 is 10 from 9 cannot be, but 10 from 19 and there remains 9, and so on as in whole Numbers, until all be finished; and the Remainder will be 179*l.* 14*s.* 10*d.*

This Example being a little considered will render all others in this Rule easy.

Examples in Weight.

Troy Weight.

	℔	oz.	pwt.	gr.
From	9	10	16	18
Take	5	09	18	22
	4	00	17	20

Averdupois Weight.

c.	qr.	℔	oz.
17	2	15	10
14	3	18	12
2	2	24	14

Examples in long Measure.

	yards	qrs.	nails
From	78	3	2
Take	29	3	3
Rests	48	3	3

miles	fur.	pole	yds.	feet	inches.
22	3	26	3½	0	9
18	6	29	4	2	11
3	4	36	4.	0	10

Example in Time.

	days	°	′	″
From	27	18	35	21
Substract	16	21	46	36
Remains	10	20	48	45

The Proof of Addition and Substraction in these Numbers of different Denominations, is the very same with that of whole Numbers in *page* 13. I shall therefore refer you to that Place, and omit repeating it here.

Sect. 4. *Of* Reduction.

BY Reduction, Numbers of different Denominations are brought into one Denomination.

That is, it alters or changes any superior Denomination proposed into any inferior or lesser Denomination required; still keeping

keeping them equivalent in Value. And by that Means they become fitly prepared for Multiplication and Division; which otherwise could not so conveniently be performed. Therefore the Business of Reduction is very useful in the Rule of Proportion, commonly called the *Golden Rule*, or *Rule of Three*, especially to those who do not understand either Vulgar or Decimal Fractions: And it is thus performed.

RULE.

Consider how many Units of the Denomination required, make one of that Denomination proposed to be reduced, which is easily known by its respective Table, and with that Number of Units, multiply the Denomination proposed, and their Product will be the Number required.

Example in Coin.

Let it be required to reduce or change 357*l.* into Shillings, and those Shillings into Pence, which shall still be equal in Value with the 357*l.*

```
                  357
Multiply with      20 the Shillings in one Pound.
                 ----
                 7140=the Shillings in 357l.
Multiply with    12     the Pence in one Shilling.
                 ----
                 1428
                 714
                 -----
                 85680= the Pence in 357l. as was required.
```

Or 357*l.* may be reduced into Pence, at one Operation: Thus,

```
                 357l.
Multiply with    240 the Pence contained in one Pound.
                 ----
                 1428
                 714
                 -----
                 85680= the Pence in 357l. as before.
```

But when the Numbers proposed to be reduced are of several Denominations, and it is required to bring them all to the lowest; you must reduce the highest or greatest Denomination to the next less, adding the Numbers that are of that less Denomination together, then reduce their Sum to the next lower Denomination, adding together all the Numbers that are of that Denomination, and so proceed gradually on unti all is done.

EXAMPLE.

Let it be required to reduce 375*l.* 17*s.* 10*d.* 3*q.* into one Denomination, *viz.* into Farthings.

375*l.* 17*s.* 10*d.* 3*q.*
20

7500= the Shillings in 375*l.*
+ 17*s*

7517= the Shillings in 375*l.* 17*s.*
12

15034
7517

90204= the Pence in 375*l.* 17*s.*
+ 10*d.*

90214= the Pence in 375*l.* 17*s.* 10*d.*
4

360856= the Farthings in 375*l.* 17*s.* 10*d.*
+ 3*q.*

360859 Farthings=375*l.* 17*s.* 10*d.* 3*q.* as was required.

The Work of this Example, and all other Operations of this Kind, may be somewhat shortned by observing the following Method.

375*l.* 17*s.* 10*d.* 3*q.*
20 Multiply and add in the 17*s.*

7517
12 Multiply and add in the 10*d.*

15034
7518

90214
4 Multiply and add in the 3*q.*

360859 the Farthings as before.

Examples in Troy Weight.

Suppose it be required to reduce 29 lb 8 *oz.* 18 *pwt.* 21 *gr.* into the least Denomination, *viz.* into Grains.

Thus,

Thus,	29 lb. 8 *oz.* 18 *pwt.* 21 *gr.*	
Multiply with	12	the *oz.* in 1 lb. and add in the 8 *oz.*
	66	
	29	
	356	=the Ounces in 29 lb. 8 *oz.*
Multiply with	20	the *pwts.* in 1 *oz.* and add in the 18 *pwt.*
	7138	=the *pwts.* in 29 lb. 8 *oz.* 18 *pwt.*
Multiply with	24	the Grains in 1 *pwt.* and add in the 21 *gr.*
	28553	
	14278	
	171333	the Grains=29 lb. 8 *oz.* 18 *pwt.* 21 *gr.*

These two Examples at large being well underſtood, may ſuffice to ſhew how all Operations of this Kind are performed; either in Weights, Meaſures, or Time. I ſhall only inſert a few Examples of each Sort for the Learner's Practice.

1. In 23 *C.* 3 *qrs.* 21 lb. 9 *oz.* Averdupois Weight; How many Ounces? *Anſwer.* 42905 Ounces.

2. In 252 *Engliſh* Miles, How many Yards, Feet and Inches? *Anſwer.* 443520 Yards=1330560 Feet=15966720 Inches.

3. In 1692 common Years, How many Days, Hours and Minutes? *Anſwer.* 618003 Days, 14832072 Hours, 889924320 Minutes.

Note, a common Year=365 Days, 6 Hours, ſee *page* 37.

4. In 5786 Pounds, 17 Shillings, 9 Pence Sterling: How many Shillings, Pence and Farthings?

Anſwer. 115737 Shillings, 1388853 Pence, or 5555412 Farthings. That is, 5786*l.* 17*s.* 9*d.*=115737*s.* 9*d.*=1388853*d.* &c.

The next Thing will be to ſhew how to bring Numbers from a leſſer to a greater Denomination, which by moſt Authors is called, tho' very improperly,

Reduction *Aſcending.*

This Work is the Converſe of the laſt, and is performed by Diviſion. Thus,

RULE.

Conſider how many of the Denomination propoſed make one of the Denomination required, and make that Number your Diviſor, by which divide the Denomination propoſed; and the Quotient will be the Number required.

EX-

EXAMPLE.

Let it be required to find how many Shillings and Pounds are contained in 85680 Pence.

The Pence in 1*s*. are 12) 85680 (7140*s*=85680*d*.

Again the Shillings in 1*l*. are 20) 7140 (357*l*. the Anſwer required.

Another Example in Coin.

How many Pence, Shillings and Pounds, are contained in 264859 Farthings.

```
        12)          20)
4) 264859 ( 66214d. (5517s. 275l.
   ------   ------   -----
   24       62       151
   ------   ------   -----
   08       21       117
   ------   ------   -----
    05        94     (17)s.
   ------   ------
      19    (10)d.
   ----------
```

Remains (3)*q*. { *Note*, The Remainder is always of the ſame Denomination with the Dividend.

The laſt Quotient 275*l*. together with the ſeveral Remainders, gives the Anſwer required.

Viz. 275*l*. 17*s*. 10*d*. 3*q*.=264859 Farthings.

Example in Troy Weight.

Suppoſe it were required to find how many *pwts*. *ozs*. and lb*s*. are contained in 171333 Grains.

```
              20)        12)
24) 171333 gr. (7138 pw. (356 (29 lb.
    168...      113        24
    ------     -----     -----
     33         138       116
     24        -----      108
    ------   (18) pwt.   -----
     93                  (8) oz.
     72
    ------
     213
     192
    ------
Remains (21) gr.
```

Anſwer, 29 lb. 8 *oz*. 18 *pwt*. 21 *gr*. This and the laſt Example are the Reverſe or Proof of thoſe in *Pages* 43, 45.

1. In 42905 Ounces Averdupois Weight; How many Pounds, &c.

Thus

```
                   28)           4)
Thus  16) 42905   (2681 lb.    (95 qrs.  (23 C.
          -----   ---------    ------
           109      252          15
          -----   ---------    ------
           130      161          (3)
          -----     140
             25   ---------
          -----
            (9)     (21) Answer, 23 C. 3 qrs. 21 lb. 9 oz.
```

2. In 15966720 Inches; How many *English* Miles, &c. *Answer*, 252 Miles, &c. As Occasion requires.

There are many useful Questions may be answered by Help of Reduction only: As the changing of one Sort of Coin for another; and comparing of one Sort of Measure with another, &c.

For Instance; Suppose one had 347 Rix-Dollars, at 4*s.* 6*d.* *per* Dollar; and desired to know how many Pounds *Sterling* they make.

```
        347
         54=the Pence in one Dollar, viz. 4s. 6d.=54d.
       -----
       1388
       1735  (20
       -----
   12) 18738d.  (1561s.  (78l.
   -----------  -------
         67       161
       -----    -------
         73       (1)s.
       -----
         18
       -----
        (6)d.
```

Answer, 78*l.* 1*s.* 6*d.* *Sterling*, are=347 Rix-Dollars.

Quest. 2. In 645 *Flemish* Ells; How many Ells *English*?

Note, 3 Quarters of a Yard *English*, make one Ell *Flemish*, an $1\frac{1}{4}$ or 5 Quarters of a Yard, is an *English* Ell.

```
          Therefore, 645
                       3=the qrs. of a Yard in 1 Ell Flemish.
                   -------
qrs. in 1 Ell=5) 1935 (387 English Ells for the Answer.
```

Quest. 3. Suppose a Bill of Exchange were accepted at *London*, for the Payment of 400*l.* *Sterling*, for the Value deliver'd at *Amsterdam* in *Flemish* Money at 1*l.* 13*s.* 6*d.* for 1*l.* *Sterling*. How much *Flemish* Money was delivered at *Amsterdam*?

First. 1*l.* 13*s.* 6*d.*=402*d.* the Value of one Pound *Sterling* at *Amsterdam*.

Then

Then $402d.\times400=160800d.=670l.$ *Flemish*, and so much was deliver'd at *Amsterdam*.

CHAP. IV.

Of Vulgar Fractions.

Sect. 1. *Of* Notation.

A Fraction, or broken Number, is that which represents a Part or Parts of any Thing proposed (*vide page* 3.) and is expressed by two Numbers placed one above the other with a Line drawn betwixt them.

Thus $\left\{\begin{array}{l}3 \text{ \textit{Numerator.}} \\ \overline{4} \text{ \textit{Denominator.}}\end{array}\right.$

The Denominator or Number placed underneath the Line, denotes how many equal Parts the Thing is supposed to be divided into (being only the Divisor in Division.) And the Numerator or Number placed above the Line, shews how many of those Parts are contained in the Fraction, it being the Remainder after Division. (See *page* 29.) And these admit of three Distinctions:

viz. $\left\{\begin{array}{l}\textit{Proper} \text{ or } \textit{Simple,} \\ \textit{Improper,} \\ \textit{Compound,}\end{array}\right\}$ *Fractions.*

A proper, pure, or simple Fraction, is that which is less than an Unit. That is, it represents the immediate Part or Parts of any Thing less than the Whole, and therefore its Numerator is always less than the Denominator.

As $\left\{\begin{array}{l}\frac{1}{4} \text{ is one \textit{Fourth Part.}} \\ \frac{1}{3} \text{ is one \textit{Third Part.}}\end{array}\right.$ And $\left\{\begin{array}{l}\frac{1}{2} \text{ is one \textit{Half.}} \\ \frac{2}{3} \text{ is two \textit{Thirds}, \&c.}\end{array}\right.$

An improper Fraction is that which is greater than an Unit; That is, it represents some Number of Parts greater than the whole Thing; and its Numerator is always greater than the Denominator.

As $\frac{5}{3}$ Or $\frac{9}{7}$ Or $\frac{41}{17}$ &c.

A compound Fraction is a Part of a Part, consisting of several Numerators and Denominators, connected together with the Word [of].

As $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$, &c. and are thus read, The one Third of the three Fourths of the two Fifths of an Unit.

That is, when a Unit (or whole Thing) is first divided into any Number of equal Parts, and each of those Parts are subdivided

divided into other Parts, and so on: Then those last Parts are called compound Fractions, or Fractions of Fractions.

As for Instance, suppose a Pound *Sterling*, or 20*s*. be the Unit or Whole; then is 8*s*. the $\frac{2}{5}$ of it, and 6*s*. the $\frac{3}{4}$ of those two Fifths, and 2*s*. is the $\frac{1}{3}$ of those three Fourths. *Viz.* 2*s*.$=\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$ of one Pound *Sterling*.

All compound Fractions are reduced to single ones, Thus,

RULE.

Multiply *all the* Numerators *into one another for a* Numerator, *and all the* Denominators *into one another for the* Denominator.

Thus the $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$ will become $\frac{6}{60}$. Or $\frac{1}{10}$.
For $1\times3\times2=6$ the Numerator, and $3\times4\times5=60$ the Denominator, but $\frac{6}{60}$ or $\frac{1}{10}$ of a Pound *Sterling* is 2*s*. As above.

Sect. 2. *To* Alter *or* Change *different* Fractions *into one* Denomination *retaining the same Value.*

IN order to gain a clear Understanding of this Section, it will be convenient to premise this Proposition, *viz.* "If a Number multiplying two Numbers produce other Numbers, the "Numbers produced of them shall be in the same Proportion "that the Numbers multiplied are, 17 *Euclid* 7.

That is to say, If both the Numerator and Denominator of any Fraction be equally multiplied into any Number, their Products will retain the same Value with that Fraction.

As in these, $\frac{2\times2}{3\times2}=\frac{4}{6}$. Or $\frac{2\times3}{3\times3}=\frac{6}{9}$. Or $\frac{2\times5}{3\times5}=\frac{10}{15}$, &c.

That is, $\frac{2}{3}$ and $\frac{4}{6}$. Or $\frac{2}{3}$ and $\frac{6}{9}$. Or $\frac{2}{3}$ and $\frac{10}{15}$ are of the same Value in respect to the Whole or Unit.

From hence it will be easy to conceive how two, or more Fractions that are of different Denominations, may be alter'd or chang'd into others that shall have one common Denominator, and still retain the same Value.

Example. Let it be required to change $\frac{2}{3}$ and $\frac{3}{7}$ into two other Fractions that shall have one common Denominator, and yet retain the same Value.

According to the foregoing Proposition, if $\frac{2}{3}$ be equally multiplied with 7, it will become $\frac{14}{21}$ *viz.* $\frac{2\times7}{3\times7}=\frac{14}{21}$. Again if $\frac{3}{7}$ be equally multiplied with 3, it will become $\frac{9}{21}$ *viz.* $\frac{3\times3}{7\times3}=\frac{9}{21}$. And by this Means I have obtained two new Fractions $\frac{14}{21}$ and $\frac{9}{21}$ that are of one Denomination, and the same Value with the two first proposed, *viz.* $\frac{14}{21}=\frac{2}{3}$ and $\frac{9}{21}=\frac{3}{7}$.

And from hence doth arise the general Rule for bringing all Fractions into one Denomination.

RULE.

Multiply *all the* Denominators *into each other for a new (and common)* Denominator. *And each* Numerator *into all the* Denominators *but its own, for new* Numerators.

Example. Let the proposed Fractions be, $\frac{1}{3}$. $\frac{2}{5}$. $\frac{3}{4}$. and $\frac{6}{7}$.

Then by the *Rule.*

A new Denominator will be thus found.	And the new Numerators will be thus found.			
3	1 .	2 .	3 .	6 .
5	5	3	3	3
15	5	6	9	18
4	4	4	5	5
60	20	24	45	90
7	7	7	7	4
420	140 .	168 .	315 .	360

Hence 420 is the common Denominator. And 140 . 168 . 315 . 360 are the new Numerators, which being placed Fractionwise are $\frac{140}{420}$. $\frac{168}{420}$. $\frac{315}{420}$. $\frac{360}{420}$. the new Fractions required.

That is, $\frac{140}{420}=\frac{1}{3}$. $\frac{168}{420}=\frac{2}{5}$. $\frac{315}{420}=\frac{3}{4}$. and $\frac{360}{420}=\frac{6}{7}$.

Sect. 3. *To bring mixed* NUMBERS *into* FRACTIONS, *and the contrary.*

MIXED Numbers are brought into improper Fractions by the following

RULE.

Multiply *the* Integers *or whole* Numbers, *with the* Denominator *of the given* Fraction, *and to their Product add the* Numerator, *the Sum will be the* Numerator *of the* Fraction *required.*

Example. $9\frac{4}{5}$ by the Rule will become $\frac{49}{5}$. For $9\times5=\frac{45}{5}$.
And, $\frac{45}{5}+\frac{4}{5}=\frac{49}{5}$ the improper Fraction required.
Again, $13\frac{11}{15}$ will become $\frac{206}{15}$. For $13\times15=\frac{195}{15}$.
And $\frac{195}{15}+\frac{11}{15}=\frac{206}{15}$. And so for any other as Occasion requires.

To find the true Value of any improper Fraction given is only the Converse of this Rule. For if $\frac{49}{5}=9\frac{4}{5}$ as before is evident;

evident: Then it follows that if 49 be divided by 5, the Quotient will be $9\frac{4}{5}$. And if 206 be divided by 15 it will give $13\frac{11}{15}$, &c. consequently it follows, That

If the Numerator of any improper Fraction be divided by its Denominator, the Quotient will discover the true Value of that Fraction.

EXAMPLES.

$\frac{35}{7}=5$. And $\frac{50}{9}=5\frac{5}{9}$. And $\frac{121}{20}=6\frac{1}{20}$. Or $\frac{15}{4}=3\frac{3}{4}$, &c.

When whole Numbers are to be expressed Fraction-wise, it is but giving them an Unit for a Denominator. Thus 45 is $\frac{45}{1}$. 9 is $\frac{9}{1}$ and 25 is $\frac{25}{1}$, &c.

Sect. 4. *To* Abbreviate *or* Reduce Fractions *into their lowest or least* Denomination.

THIS is done, not out of any Necessity, but for the more convenient managing of such Fractions as are either proposed in large Terms; or swell into such, either by Addition or otherwise: Besides, 'tis most like an Artist to express or set down all Fractions in the lowest Terms possible; and to perform that, it will be necessary to consider of these following Propositions.

Numbers are either Prime or Composed.

1. A Prime Number is that which can only be measured by an Unit. *Euclid* 7. *Defi.* 11.

That is, 3. 5. 7. 11. 13. 17. &c. are said to be Prime Numbers, because it is not possible to divide them into equal Parts by any other Number but Unity or 1.

2. Numbers Prime the one to the other, are such as only an Unit doth measure, being their common Measure. *Euclid* 7. *Defi.* 12.

For Instance, 7 and 13 are Prime Numbers to each other, because they cannot be divided by any Number but an Unit. And 9 and 14 are also Prime Numbers to each other, for altho' 3 will measure or divide 9 without leaving a Remainder, yet 3 will not measure 14 without leaving a Remainder: Again, altho' 2 will measure 14 without any Remainder, yet 2 will not measure 9 without leaving a Remainder, &c.

3. A Composed Number is that which some certain Number measureth. *Euclid* 7. *Defi.* 13.

For Instance, 15 is a composed Number of 3 and 5. for $5\times3=15$, consequently 3 or 5 will justly measure 15. Also 20

is composed of 5 and 4, *viz.* $5\times4=20$, therefore 5 and 4 will each justly measure 20.

4. Numbers composed the one to the other, are they which some Number being a common Measure to them both doth measure. *Euclid* 7. *Def.* 14.

That is, If two or more Numbers can be divided by one and the same Divisor; then are those Numbers said to be composed one to another.

For Instance, 14 and 21 are Numbers composed the one to the other, because they can both be measured or divided by 7. For $7\times2=14$, and $7\times3=21$; therefore 7 is a common Measure to 14 and 21. So that if $\frac{14}{21}$ were proposed to be abbreviated, it will become $\frac{2}{3}$.

Thus $\left\{ \begin{array}{l} 7)\frac{14}{21} = \frac{2}{3} \\ 7) \end{array} \right.$

And how those greatest common Measures may be found, comes from *Euclid* 7. *pro.* 1. 2 3. and is thus:

RULE.

Divide the greater *Number* by the lesser, and that *Divisor* by the *Remainder* (if there be any) and so on continually until there be no *Remainder* left: Then will this last *Divisor* be the greatest Common *Measure* (and if it happen to be 1, then are those *Numbers* Prime *Numbers*, and are already in their lowest Terms, but if otherwise) *Divide* the *Numbers* by that last *Divisor*, and their *Quotient* will be their least Terms required.

EXAMPLE.

Let it be required to find the greatest common Measure of 72 and 108, *viz.* Of $\frac{72}{108}$.

```
72) 108 (1
    72
  -------
    36) 72 (2   { Her because there's no Remainder;
        72      { 36 is the greatest common Measure.
       -------
        (0)
```

Therefore, $\left\{ \begin{array}{l} 36)\frac{72}{108} = \frac{2}{3} \\ 36) \end{array} \right.$ $\left\{ \begin{array}{l} \text{Hence } \frac{72}{108} \text{ is abbreviated} \\ \text{to } \frac{2}{3} \text{ the lowest Terms.} \end{array} \right.$

Again, to find the greatest common Measure of 744 and 899.

Thus

Thus, 744) 899 (1.
744
---
155) 744 (4
620
---
124) 155 (1
124
---
31) 124 (4
124
---
(0)

Here 31 is found to be the greatest common Measure by which 744 and 899 may be abbreviated to 24 and 29 their lowest Terms.

Thus, $\frac{31}{31}$) $\frac{744}{899}$ (= $\frac{24}{29}$, &c.

Note, If the proposed Numbers be even, they may be brought lower by a continual Halving of them, so long as they can be halved, *viz.* divided by 2.

EXAMPLE.

'Tis required to reduce $\frac{56}{84}$ to its least Terms.

First, $\frac{2}{2}$) $\frac{56}{84}$ (= $\frac{28}{42}$. Again $\frac{2}{2}$) $\frac{28}{42}$ (= $\frac{14}{21}$.

This done, you may easily perceive that 7 will be the common Measure to 14 and 21, *viz.* $\frac{7}{7}$) $\frac{14}{21}$ (= $\frac{2}{3}$, &c.

If the Numbers proposed to be reduced have each a Cypher or Cyphers annexed to them, they will be abbreviated by cutting off a like Number of Cyphers from both.

Thus, $\frac{150}{300}$ will be $\frac{15}{30}$, And $\frac{200}{300}$ will be $\frac{2}{3}$, &c.

That is, $\frac{150}{300}=\frac{15}{30}=\frac{1}{2}$. And $\frac{200}{300}=\frac{2}{3}$. Also $\frac{360}{400}=\frac{36}{40}=\frac{18}{20}=\frac{9}{30}$.

Sect. 5. ADDITION *of* FRACTIONS.

WHAT hath been done by the Rules in this Chapter; is chiefly to prepare and fit Fractions of different Denominations for Addition or Substraction, as Occasion requires, *viz.* If they are compound Fractions, they must be reduced to simple or pure Fractions, *per Rule, Sect.* 1.

If they are of different Denominations, they must be altered or changed, *per Rule, Sect.* 2.

That is, all Fractions must be brought into one Denomination before they can either be added, or substracted, and that being done, Addition is thus performed.

RULE.

RULE.

Add together all the *Numerators*, and their *Sum* will be a *New Numerator*, under which *ſubſcribe* the *common Denomination*.

Examples in SIMPLE FRACTIONS.

Let it be propoſed to add $\frac{1}{3}$, $\frac{2}{5}$ and $\frac{3}{4}$ together. Firſt $\frac{1}{3}=\frac{20}{60}$. $\frac{2}{5}=\frac{24}{60}$, and $\frac{3}{4}=\frac{45}{60}$. *per Sect.* 2.

Then $\frac{20}{60}+\frac{24}{60}+\frac{45}{60}=\frac{80}{60}$, the Sum required, which according to *Section* 3. is $1\frac{20}{60}$. *viz.* $\frac{80}{60}=1\frac{20}{60}$.

Examples in COMPOUND FRACTIONS.

Let it be required to add $\frac{3}{7}$ and $\frac{2}{3}$ of $\frac{3}{4}$ into one Sum. Firſt $\frac{2}{3}$ of $\frac{3}{4}$ becomes $\frac{6}{12}$ or $\frac{1}{2}$ *per Sect.* 1 And (*per Sect.* 2.) $\frac{3}{7}$ and $\frac{1}{2}$ is $\frac{6}{14}$ and $\frac{7}{14}$, *viz.* $\frac{3}{7}=\frac{6}{14}$ and $\frac{1}{2}=\frac{7}{14}$ but $\frac{7}{14}+\frac{6}{14}=\frac{13}{14}$ the Sum required, *viz*, $\frac{3}{7}+\frac{2}{3}$ of $\frac{3}{4}=\frac{13}{14}$.

Examples in MIXED NUMBERS.

'Tis required to add $5\frac{2}{3}$ to $7\frac{3}{4}$. theſe *per Sect.* 3. will be $\frac{17}{3}$ and $\frac{31}{4}$ But $\frac{17}{3}$ and $\frac{31}{4}$ will become $\frac{68}{12}$ and $\frac{93}{12}$ *per Sect.* 2.
Then $\frac{68}{12}+\frac{93}{12}=\frac{161}{12}=13\frac{5}{12}$. the Sum required.

Or you may bring only the Fractions to one Denomination, Thus, $5\frac{2}{3}$ and $7\frac{3}{4}$ will become $5\frac{8}{12}$ and $7\frac{9}{12}$.

Then $5\frac{8}{12}+7\frac{9}{12}=12\frac{17}{12}$. That is, $13\frac{5}{12}$. As before.

Sect. 6. SUBSTRACTION *of* FRACTIONS.

RULE.

SUBSRACT one Numerator from the other (according as the Queſtion requires) and their Difference will be a new *Numerator*, under which *ſubſcribe* the common *Denominator* as in Addition.

EXAMPLE 1.

Let it be required to take $\frac{2}{9}$ out of $\frac{3}{7}$. Firſt $\frac{2}{9}$ and $\frac{3}{7}$. *per Sect.* 2. will become $\frac{14}{63}$ and $\frac{27}{63}$. then $\frac{27}{63}-\frac{14}{63}=\frac{13}{63}$, that is, $\frac{3}{7}-\frac{2}{9}=\frac{13}{63}$. As was required.

EXAMPLE 2.

'Tis required to ſubſtract $\frac{2}{3}$ of $\frac{8}{9}$ from $\frac{13}{14}$. Firſt $\frac{2}{3}$ of $\frac{8}{9}=\frac{16}{27}$. *per Sect.* 1. Again $\frac{16}{27}$ and $\frac{13}{14}$ will become $\frac{224}{378}$ and $\frac{351}{378}$ *per Sect.* 2. Then $\frac{351}{378}-\frac{224}{378}=\frac{127}{378}$.

EXAMPLE 3.

From $6\frac{1}{8}$ ſubſtract $3\frac{19}{48}$. Firſt, $6\frac{1}{8}=\frac{49}{8}$. and $3\frac{19}{48}=\frac{163}{48}$. *per Rule Sect.* 3. Again, $\frac{49}{8}=\frac{2352}{384}$, and $\frac{163}{48}=\frac{1304}{384}$. *per Rule Sect.* 2. Then $\frac{2352}{384}-\frac{1304}{384}=\frac{1048}{384}=2\frac{280}{384}=2\frac{35}{48}$. Or otherwiſe thus: Firſt, $6\frac{1}{8}=5\frac{9}{8}$, then bring $\frac{9}{8}$ and $\frac{19}{48}$ into one Denomination, *viz.* $5\frac{9}{8}=5\frac{432}{384}$ and $3\frac{19}{48}=3\frac{152}{384}$.

Then

Then $5\frac{432}{384} - 3\frac{152}{384} = 2\frac{280}{384} = 2\frac{35}{48}$. As before.

EXAMPLE 4.

Let it be required to ſubſtract $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}$ from 7.
Firſt, $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3} = \frac{30}{189}$. And $7 = 6\frac{189}{189}$.
Then $6\frac{189}{189} - \frac{30}{189} = 6\frac{159}{189} = 6\frac{53}{63} = 7 - \frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}$. As was required.

If theſe few Examples be well underſtood, the whole Buſineſs of adding and ſubſtracting Vulgar Fractions will be eaſy; which is really much more difficult to perform than either Multiplication or Diviſion, as will appear in the next *Section*.

Sect. 7. *Of* MULTIPLICATION *of* FRACTIONS.

IN order to perform either Multiplication or Diviſion, you muſt prepare the Terms to be multiplied (or divided) thus; reduce compound Fraction to ſimple ones, *per Sect.* 1. Bring mix'd Numbers into improper Fractions, and expreſs whole Numbers Eraction-wiſe, *per Sect* 3. Alſo it will be convenient to abbreviate them to their ſmalleſt Terms when it can be done. Then Multiplication may be thus performed.

RULE.

Multiply the *Numerators* one into another for a new *Numerator*; and the *Denominators* into one another, for a new *Denominator*. As in theſe

EXAMPLES.

1. The Product of $\frac{2}{5}$ into $\frac{3}{7} = \frac{6}{35}$. That is, $\frac{2\times3}{5\times7} = \frac{6}{35}$.

2. And the Product of $\frac{9}{16}$ into $\frac{20}{27} = \frac{180}{432}$. Or $\frac{5}{12}$.
3. Again, the Product of $\frac{7}{11}$ into $\frac{2}{5}$ of $\frac{5}{7} = \frac{70}{385}$. Or $\frac{2}{11}$.
For $\frac{2}{5}$ of $\frac{5}{7} = \frac{10}{35}$. Then $\frac{7}{11} \times \frac{10}{35} = \frac{70}{385} = \frac{2}{11}$.
4. Let it be required to multiply 6 with $3\frac{2}{5}$. Theſe prepared for the Work will ſtand thus. $\frac{6}{1} \times \frac{17}{5}$.
viz. $6 = \frac{6}{1}$ and $3\frac{2}{5} = \frac{17}{5}$. Then $\frac{6}{1} \times \frac{17}{5} = \frac{102}{5}$. or $20\frac{2}{5}$.
Or otherwiſe thus, $6\times3 = 18$. And $\frac{2}{5}\times6 = \frac{12}{5} = 2\frac{2}{5}$.
Then $18 + 2\frac{2}{5} = 20\frac{2}{5}$. As before.
Let it be required to multiply $7\frac{4}{9}$ with $5\frac{3}{7}$.
Firſt $7\frac{4}{9} = \frac{67}{9}$, and $5\frac{3}{7} = \frac{38}{7}$. Then $\frac{67}{9} \times \frac{38}{7} = \frac{2546}{63} = 40\frac{26}{63}$.

Now the Reaſon of this Rule for multiplying of Fractions, and conſequently of theſe Operations, and all other performed by it; will be evident from this following.

Viz.

Viz. If $\frac{4}{2}$ be multiplied with $\frac{12}{3}$ according to the Rule, their Product will be $\frac{48}{6}$. But $\frac{48}{6}=8$.
Now $\frac{4}{2}=2$. and $\frac{12}{3}=4$ *per Sect.* 3. But $4\times2=8$. *Ergo*, &c.

Sect. 8 Division *of* Fractions.

THE Fractions being first prepared as before directed, Division may be thus performed:

RULE.

Multiply the *Numerator* of the *Dividend* into the *Denominator* of the *dividing Fraction* for a new *Numerator*: And Multiply the other *Numerator* and *Denominator* together for a new *Denominator*.

EXAMPLES.

1. Let $\frac{6}{35}$ be divided by $\frac{3}{7}$ *viz.* $\frac{3}{7}$) $\frac{6}{35}$ ($\frac{42}{105}=\frac{2}{5}$ the Quotient.
That is, according to the Rule $6\times7=42$ the new Numerator, and $35\times3=105$, the new Denominator, &c. as above.
2. Let it be requir'd to divide $\frac{20}{27}$ by $\frac{5}{12}$. *viz.* $\frac{5}{12}$) $\frac{20}{27}$ ($\frac{240}{135}=1\frac{7}{9}$.
For $12\times20=240$ the new Numerator, and $27\times5=135$ the new Denominator, &c. as before.

3. Suppose it were required to divide $\frac{2}{11}$ by $\frac{2}{5}$ of $\frac{5}{7}$.
First, $\frac{2}{5}$ of $\frac{5}{7}=\frac{10}{35}$) $\frac{2}{11}$ ($\frac{70}{110}=\frac{7}{11}$.
4. Let $20\frac{2}{5}$ be divided by $3\frac{2}{5}$ *viz.* $\frac{102}{5}$ by $\frac{17}{5}$.
For $20\frac{2}{5}=\frac{102}{5}$, and $3\frac{2}{5}=\frac{17}{5}$, Then $\frac{17}{5}$) $\frac{102}{5}$ (=6 the Quotient.
5. Let it be required to divide $40\frac{26}{63}$ by $5\frac{3}{7}$.
First, $40\frac{26}{63}=\frac{2540}{63}$, and $5\frac{3}{7}=\frac{39}{7}$. Then $\frac{38}{7}$) $\frac{2546}{63}$ ($\frac{17822}{2394}$.
But $\frac{17822}{2394}=7\frac{4}{9}$ the true Quotient required.
6. Suppose it were required to divide 13 by $\frac{5}{7}$.
First, $15=\frac{13}{1}$. Then $\frac{5}{7}$) $\frac{13}{1}$ ($\frac{91}{5}=18\frac{1}{5}$, the Quotient.
7. Again, let it be required to divide $\frac{5}{7}$ by 6.
Viz. $\frac{6}{1}$) $\frac{5}{7}$ ($\frac{5}{42}$ for the Quotient required.

N. B. From hence you may observe, that when any Whole Number is divided by a Fraction less than Unity or 1, the Quotient will be greater than the Number propos'd to be divided: But if any Fraction be divided by a whole Number, greater than 1, then the Quotient will be less than the the Dividend: As in the two last Examples.

As

As to the *Reason* (or *Proof*) of this *Rule* for *dividing Fractions:* It is only the *Converse* to that of *Multiplication*, and will be very evident from this following.

Let $\frac{48}{8}$ be *divided* by $\frac{4}{2}$. Which according to the *Rule* is thus, $\frac{4}{2}$) $\frac{48}{8}$ ($\frac{96}{24}$=4. The true *Quotient*. Now $\frac{48}{8}$=8. And $\frac{4}{2}$=2. *per Sect.* 3. Consequently $\frac{48}{8}$ *divided* by $\frac{4}{2}$ is but the same with 8 *divided* by 2. *viz.* 2) 8 (4. The *Quotient* as before.

I could have inserted *Geometrical Demonstrations*, for the *Rules* of *Multiplication* and *Division* of *Fractions*; but supposing the Learner purely unacquainted with those Kind of Demonstrations, I thought these might be more intelligible to him, especially in this Place.

CHAP. V.

Of Decimal Fractions.

WHEN, *or by whom, this excellent Invention of* Decimal Arithmetick *was first introduced, is uncertain; but doubtless its Improvements, and the Perfections it is now in, is owing to latter Years.*

Sect. 1. *Of* Notation.

IN *Decimal Fractions*, the *Integer* or *whole Thing* (whether it be *Coin*, *Weight*, *Measure*, or *Time*, &c.) is supposed to be *divided* into *Ten equal Parts*; and every one of those *Ten Parts* are supposed to be *subdivided* into other *Ten equal Parts*, &c. *ad infinitum.*

The *Integer* being thus *divided (by Imagination)* into 10, 100, 1000, 10000, &c. *equal Parts*, becomes the *Denominator* to the *Decimal Fractions*.

thus, $\frac{2}{10}$. $\frac{3}{100}$. $\frac{7}{1000}$. $\frac{53}{10000}$. $\frac{754}{100000}$, &c.

Now these *Denominators* are seldom or never set down, but only the *Numerators*, and those are either distinguished, or separated from *whole Numbers* by a *Point* or a *Comma*.

Thus, 5,4 is 5$\frac{1}{10}$. and 0,7 is $\frac{7}{10}$. 35,05 is 35 $\frac{5}{100}$, &c.

But before we proceed further in *Notation*, it will be convenient for the Learner to consider the following *Table*, (taken out of the learned Mr. *Oughtaed*'s *Clavis Mathematica*) which shews the very Foundation of *Decimal Fractions*.

Whole Numbers,						Decimal Parts.					
5 &c.	4 Tens of Thousands.	3 Thousands.	2 Hundreds.	1 Tens.	0, Units Place.	1 Part of Ten, or $\frac{1}{10}$.	2 Parts of a Hundred.	3 Parts of a Thousand.	4 Parts of Ten Thousand.	5 Parts of 100 Thousand.	6 Parts of a Million. &c.

By this *Table* it is evident, that as in whole *Numbers* or *Integers*, every *Degree* from the *Unit*'s Place increases towards the Left-hand by a Ten-fold *Proportion:* So in *Decimal Parts* every *Degree* is decreased towards the Right-hand by the same *Proportion*, viz. by *Tens*.

Therefore these *Decimal Parts* or *Fractions*, are really more *Homogeneal*, or agreeing with *whole Numbers*, than *Vulgar Fractions*; for indeed all plain *Numbers* are in effect but *Decimal Parts* one to another.

That is, suppose any *Series* of *equal Numbers*, as 444, &c. The first 4 towards the Left is *Ten* times the *Value* of the 4 in the Middle, and that 4 in the Middle is *Ten* times the *Value* of the last 4 to the Right of it, and but the *Tenth Part* of that 4 on the Left, &c.

Therefore all or any of them may be taken either as *Integers*, or *Parts* of an *Integer:* If *Integers*, then they must be set down without any *Comma* or separating *Point* betwixt them thus, 444. But if *Integers*, and one *Part* or *Fraction*, put a *Comma* betwixt them thus, 44,4 which signifies 44 *whole Numbers*, and 4 *Tenths* of an *Unit:* Again, if two Places of *Parts* be required, separate them with a *Comma* thus, 4,44 *viz.* 4 *Units*, and 44 *hundred Parts* of an *Unit*, &c.

From hence (duly compared with the *Table*) it will be easy to conceive that *Decimal Parts* take their *Denomination* from the Place of their last *Figure*.

That is, $\left\{\begin{array}{l} ,5=\frac{5}{10} \\ ,56=\frac{56}{100} \\ ,056=\frac{56}{1000} \end{array}\right\}$ *Parts* of an *Unit*, &c.

Cyphers

Cyphers annexed to *Decimal Parts*, alter not their *Value*. As ,50, and ,500, or ,5000, *&c.* are each but 5 *Tenths* of an *Unit*, For $\frac{50}{100}=\frac{5}{10}$. And $\frac{500}{1000}=\frac{5}{10}$. Or $\frac{5000}{10000}=\frac{5}{10}$. *Per Sect.* 4. of the last *Chapter*.

But *Cyphers* prefixed to *Decimal Parts* decrease their *Value*, by removing them further from the *Comma*.

Thus, $\left\{\begin{array}{l} ,5=5 \textit{ Tenth Parts.} \\ ,05=5 \textit{ Parts of a Hundred.} \\ ,005=5 \textit{ Parts of a Thousand.} \\ ,0005=5 \textit{ Parts of Ten Thousand, \&c.} \end{array}\right.$

Consequently the true *Value* of all *Decimal Parts* are known by their Distance from the *Units* Place; the which being once rightly understood, the rest will be easy.

Sect. 2. Addition *and* Substraction *of* Decimals.

IN setting down the proposed Numbers to be added, or substracted, great Care must be taken in placing every Figure directly underneath those of the same Value whether they be mix'd Numbers, or pure Decimal Parts, and to perform that you must have a due Regard to the Comma's, or separating Points, which ought always to stand in a direct Line one under another; and to the Right-hand of them carefully place the Decimal Parts, according to their respective Values, or Distances from Unity. Then

Rule $\left\{\begin{array}{l} \textit{Add or substract them, as if they were all whole Numbers;} \\ \textit{and from their Sum, or Difference, cut off so many Deci-} \\ \textit{mal Parts as are the most in any of the given Numbers.} \end{array}\right.$

EXAMPLES in Addition.

Let it be required to find the Sum of these following Numbers, *viz.* 34,5+65,3+128,7+95+87,8+7,9, which being truly placed, will stand

Thus $\left\{\begin{array}{r} 34,5 \\ 65,3 \\ 128,7 \\ 95,0 \\ 87,8 \\ 7,9 \\ \hline \end{array}\right.$

Their Sum required, 419,2

EXAMPLE 1.

Let it be required to find the Sum of 25,854+34,578+9,076 +13,907.

25,854
34,578
9,076
13,907
83,415 The Sum required.

When the Decimal Parts proposed to be added (or substracted) have not the same Number of Places, you may for convenience of Operation supply or fill up the void Places, by annexing Cyphers. As in these *Examples*.

EXAMPLE 3.	*EXAMPLE* 4.	*EXAMPLE* 5.
45,0700	574,678953	0,975642
50,7580	95,796430	,745257
123,0057	78,054600	,000598
74,7020	54,789000	,800700
24,8000	8,900000	,640530
318,3357	Sum 812,218983	3,162727

EXAMPLES in SUBSTRACTION.

Let it be required to find the Difference between 45,375 and 74,284.

EXAMPLE 1.	*EXAMPLE* 2.	*EXAMPLE* 3.
That is, From 74,284	From 437,5	From 75,0034
Take 45,375	Take 89,657	Take 57,875
Remains 28,909	347,843	17,1284

EXAMPLE 4.

Let it be required to find the Excess between 562 and 93,5784.

EXAMPLE 4.	*EXAMPLE* 5.
That is, From 562,	From 345,7578
Take 93,5784	Take 157,
The Excess 468,4216	188,7578

Note, The two last Examples are supposed to be supplied with Cyphers, which if actually done would stand thus.

562,0000	345,7578
93,5784	157,0000
Remains 468,4216 As before,	188,7578

EXAMPLE.

EXAMPLE 6.	*EXAMPLE 7.*
From 0,547893	From 1,000000
Take 0,439758	Take 0,997543
0,108135	0,002457

The Proof of Addition and Substraction in Decimals, is the same with that of whole Numbers, Page 13, &c.

Sect. 3. Multiplication *of* Decimals.

WHETHER the Factors or Numbers to be multiplied are pure Decimals, or mixed. Multiply them as if they were all whole Numbers, and for the true Value of their Product observe this

Rule. { *Cut off* (viz. *separate with a Comma*) *so many Places of Decimal Parts in the Product, as there are in both the Factors accounted together. As in these,*

EXAMPLE 1.	*EXAMPLE 2.*
3,024	32,12
2,23	24,3
9072	9636
6048	12848
6048	6424
6,74352	780,516

The Reason why such a Number of Decimal Parts must be cut off in the Product, may be easily deduced from these Examples. Thus,

In Example 1. It is evident, that 3, the whole Number in the Multiplicand, being multiplied with 2, the whole Number in the Multiplier; can produce but 6 (*viz.* $3\times2=6$). So that of Necessity all the other Figures in the Product must be Decimal Parts; according as the Rule directs.

Or, the Rule is evident from the Multiplication of whole Numbers only: Thus, suppose 3000 were to be multiplied with 200, their Product will be 600000; That is, there will be so many Cyphers in the Product, as are in both the Factors, (*Vide page* 18). Now if, instead of those Cyphers in the Factors, we suppose the like Number of Decimal Parts; then it follows, that there ought to be the same Number of Decimal Parts in the Product, as there were Cyphers in the Factors.

Again the Rule may be otherwise made evident from Vulgar Fractions, thus: Let 32,12 be multiplied with 24,3, and

and their Product will be 780,516 as in *Example* 2, above. Now $32,12=32\frac{12}{100}$. and $24,3=24\frac{3}{10}$ which being brought into Improper Fractions (*per Sect.* 3. *page* 50,) will become $32\frac{12}{100}=\frac{3212}{100}$. and $24\frac{3}{10}=\frac{243}{10}$.

Then $\frac{3212}{100}\times\frac{243}{10}=\frac{780516}{1000}$. *per Sect.* 7. *page* 55.

But $\frac{780516}{1000}=780\frac{516}{100}$. *viz.* 780,516, as before.

Any of these three Ways do, I presume, sufficiently prove the Truth of the abovesaid Rule, &c.

EXAMPLE 3.

```
     78,546
        436
    -------
     471276
    235638
   314184
   --------
   34246,056
```

EXAMPLE 4.

```
     5745
    ,0675
   ------
    28725
   40215
  34470
  -------
  387,7875
```

N. B. *It sometimes falls out in multiplying Parts with Parts, that there will not be so many Figures in the Product, as there ought to be Places of Decimal Parts by the Rule: In that Case you must supply their Defect by prefixing Cyphers to the Product; as in these Examples.*

EXAMPLE 5.

```
     ,2365
     ,2435
    ------
     11825
     7095
    9460
   4730
  --------
  ,05758775
```

EXAMPLE 6.

```
    ,0347
    ,0236
   ------
     2082
    1041
    694
  --------
 ,00081892
```

When any proposed Number of Decimals is to be multiplied with 10 . 100 . 1000 . 10000, &c. It is only removing the separating Point in the Multiplicand, so many Places towards the Right-hand, as there are Cyphers in the Multiplier.

Thus, $,578\times10=5,78$. And $,578\times100=57,8$.

Again, $,578\times1000=578$. Or, $,578\times1000=5780$.

These

These Things being considered, it will be easy to multiply Decimals, and determine their true Products. As in these following Examples.

57,056 multiplied into 0,578 will produce 32,978368
7,6[illegible] into 5,4246 will produce 41,52151578
0,56879×0,05674=0,0322731446
0,03246×0,02364=0,0007672544
87649×0,03687=3231,61863
94,35786×6,57869=620,7511100034
3,141592×52,7438=165,6995001296

Now it oftentims happens, that it will be needless to express all the Figures of the Product at large, (especially, when the Factors have each of them many Places of Decimal Parts, as in the two last Examples) only so many of them as may suffice for the intended Design; and yet the Product may be as true to so many Figures as are retained, as if the Factors had been multiplied at large. And such compendious Contractions are not only of Curiosity, but may also be found of great Ease and Use to the ingenious Practitioner; especially in resolving adfected Equations, or in calculating of Trigonometrical Problems by the natural Sines and Tangents, &c. All which may be thus performed.

Viz. *Set the Unit's Place of the Multiplier directly underneath that Figure of the Multiplicand, whose Place you intend to keep in the Product; and place all the other Figures of the Multiplier in a quite contrary Order to the usual Way. Then in multiplying always begin at that Figure of the Multiplicand which stands over the Figure wherewith you are then a multiplying, setting down the first Figure of each particular Product, directly underneath one another; yet herein you must have a due Regard to the Increase which would arise out of the two next Figures to the Right-hand of that Figure in the Multiplicand which you then begin with.*

EXAMPLE.

Let it be required to multiply 3,141592 with 52,7438 and let there be only four Places of Decimal Parts retained in the Product.

If the proposed Numbers were to be multiplied at large they must stand in a direct Order as usual.

Thus $\left\{\begin{array}{l}3{,}141592\\ \underline{52{,}7438}\end{array}\right\}$ And would produce ten Places of Parts, as in the last Example.

But

But seeing it is required to have only four Places of those Parts in the Product, set them down as before directed, and they will stand

Thus 3,141592	The Multiplicand placed as before.
8347,25	The Multiplier in a reverse Order.
1570796	The Product with 5, regard had to 5 times 2.
62832	The Product with 2, increased with 9×2.
21991	Product with 7, increased with 5×7+9×7.
1257	Product with 4, increased with 1×4+5×4.
94	Product with 3, increased with 4×3.
25	Product with 8, increased with 4×8+1×8.
165,6995	The true Product as was required.

The Reason of this Contraction is very obvious from the whole Operation wrought at large.

Thus 3,141592
52,7438

25	132736
94	24776
1256	6368
2 1991	144
6 2831	84
157 0796	0
165,6995	001296

From hence it is evident, that all the Figrues in the Square to the Right-hand, are wholly omitted in the former Contraction; and that the last single Product *here, is the first there; consequently the Reason of placing the* Multiplier *in a reverse* Order, *must needs appear very plain.*

EXAMPLE 3.

Suppose it were required to multiply 257,356 with 76,48 and to have only the entire Product of Integers.

257,356
84,67
18015
1544
103
20
19682

The same at large { 257,356
76,48

20	58848
102	9424
1544	136
18014	92
19682	58688

The chiefest Care and Difficulty that attends these Contractions, is the true setting down of the Unit's Place in the Multiplier underneath the proper Figure of the Multiplicand, according to the designed Product.

Viz.

Viz. In Example 1. It was required to have four Places of Decimal Parts in the Product; therefore the Unit's Place of the Multiplier was set under the fourth Place of Decimals in the Multiplicand: And in Example 2, because it was required to have an entire Product of Integers only; therefore the Unit's Place of the Multiplier was set under the Unit's Place of the Mulultiplicand. This, I say, being once rightly understood, will render the Method easy in Practice.

Sect. 4. Division *of* Decimals.

DIVISION is accounted the most difficult Part of Decimal Arithmetick: In order therefore to make it plain and easy, it will be convenient to resume what has been said in *page* 25.

Viz. { *The Quotient Figure is always of the same Value or Degree with that Figure of the Dividend, under which the Unit's Place of its Product stands.*

As for Instance, Let 294 be divided by 4.

```
4) 294 (7   { This is not 7 but 70, because the Unit's
   28        { Place of 4×7 stands under the Tens Place
             { of the Dividend.
   ---
    14 (3  But this is only 3.
    12
   ---
Remains (2) Hence 73¾ is the Quotient.
```

Now if to the Remainder 2 there be annexed a Cypher (thus, 2,0) and then divided on, it must needs follow that the Units Place of the Product arising from the Divisor into the Quotient, will stand under the annexed Cypher; consequently the Quotient Figure will be of the same Value or Degree with the Place of that Cypher: But that is the next below the Unit's Place, therefore the Quotient Figure is of the next Degree or Place below Unity; That is, in the first Place of Decimal Parts.

Thus 4) 2,0 (,5

So that 4) 294,0 (73,5 the true Quotient required.

This being well understood; Division of Decimals may (in all the various Cases) be easily performed. However, that it may be rendered plain and easy even to the meanest Capacity, if possible, let Division be again defined, as in *page* 21.

Viz. *If that Number which divides another, be multiplied with the Number which is quoted, their Product will be the Number divided.*

This Definition alone (it compared with the Rule *page* 61.) will afford a general Rule for discovering the true Value of the Quotient Figure in Division of Decimals.

Rule { *The Place of Decimal Parts in the Divisor and Quotient, being counted together, must always be equal in Number with those in the Dividend. And from this general Rule ariseth four Cases.*

Case 1. When the Places of Parts in the Divisor and Dividend are equal, the Quotient will be whole Numbers.

As in these *Examples.*

```
8,45) 295,75 (35          0,0078) ,4368 (56
      253 5                       390
      -----                       ---
       42 25                      468
       42 25                      468
      ------                      ---
        (0)                       (0)
```

Case 2. When the Places of Parts in the Dividend exceed those in the Divisor; cut off the Excess for Decimal Parts in the Quotient. As in these *Examples.*

```
24,3) 780,516 (32,12                     436) 34246,056 (78,546
      729                                     3052
      ---                                     ----
       515                                     3726
       486                                     3488
       ---                                     ----
        291                                     2380
        243                                     2180
        ---                                     ----
         486     ,534) ,30438 (57,               2005
         486           2670                      1744
         ---           ----                      ----
         (0)            3738                      2616
                        3738                      2616
                        ----                      ----
                         (0                        (0)
```

Case 3. When there are not so many Places of Parts in the Dividend, as are in the Divisor; annex Cyphers to the Dividend to make them equal. Then will the Quotient be whole Numbers, as in Case 1.

EXAMPLES.

EXAMPLES.

Let it be required to divide 192,1 by 7,684, and 441 by ,7875

```
7,684) 192,100 (25          ,7875) 441,0000 (560
       153 68                      393 75
       ------                      ------
        38 420                      47 250
        38 420                      47 250
        ------                      ------
          (0                          (0)
```

Case 4. If after Division is finished, there are not so many Figures in the Quotient, as there ought to be Places of Parts by the general Rule; supply their defect by prefixing Cyphers to it.

EXAMPLES.

Let it be required to divide 7,25406 by 957.

957) 7,25406 (,00758 the true Quotient required.

```
     6699
     ----
      5550            Again ,575) ,0007475 (,0013
      4785                          575
      ----                          ----
       7656                         1725
       7656                         1725
       ----                         ----
        (0)                          (0)
```

Note, When Decimal Numbers are to be divided by 10. 100. 1000. 10000. &c. that is, when the Divisor is an Unit with Cyphers; Division is performed by removing or placing the separating Point in the Dividend, so many Places towards the Left-hand, as there are Cyphers in the Divisor.

EXAMPLE.

10) 5784 (578,4	100) 578,4 (57,84
1000) 5784 (5,784	10000) 578,4 (,5784

Note, *These Operations are the direct Converse to those in* page 62.

I presume it needless to give more Examples at large; only I shall insert a few Dividends, and Divisors, with their Quotients, wherein are contained all the Varieties that can happen in Division of Decimals.

574) 493,066 (859	5,74) 49,3066 (8,59
574) 493,066 (,859	5,74) 493066,00 (85900
574) 49,3066 (,0859	,0574) 493,0666 (8590
5,74) 4930,66 (859	,0574) ,493066 (8,59

There is alſo a compendious Way of contracting Diviſion, like that of Multiplication, *page* 64, by which much Labour may be ſaved; eſpecially when the Diviſor hath many Places of Decimal Parts in it: And it is thus performed.

Having determined how many Places of whole Numbers there will be in the Quotient, if any at all; or if none, of what Value or Place the firſt Figure in the Quotient will be: Then omit, or prick off one Figure of the Diviſor at each Operation; *viz.* for every Figure you place in the Quotient, prick off one in the Diviſor; having a due Regard to the Increaſe which would ariſe from the Figure ſo omitted.

EXAMPLE.

Let it be required to divide 70,23 by 7,9863.

```
The Work contracted.              The ſame at Length.

7,9863) 70,2300 (8,7938           7,9863) 70,2300 (8,7938
....    63 8904 |                         63 8904|
        -------                           -------------
         6 3396 |                          6 3396|0
         5 5904 |                          5 5904|1
        -------                           -------------
           7492 |                            7491|90
           7187 |                            7187|67
           ----                           -------------
            305 |                             304|230
            239 |                             239|589
            ---                           -------------
             66 |                              64|6410
             64 |                              63|8904
            ---                           -------------
            (2) |                               0|7506
```

The Work contracted I preſume is ſo obvious (if compared with the ſame at large) that it is needleſs to give any farther Explanation of it.

Sect. 5. *To* Reduce VULGAR FRACTIONS *into* DECIMALS, *and the contrary.*

ANY Vulgar Fraction being given, it may be reduced, or rather changed into Decimal Parts equivalent to it. Thus,

Rule: *Annex Cyphers to the Numerator, and then divide it by the Denominator, the Quotient will be the Decimal Parts equivalent to the given Fraction; or at leaſt ſo near it as may be thought neceſſary to approach.*

EXAMPLE.

EXAMPLE.

It is required to change or reduce $\frac{3}{4}$ into Decimals.

4) 3,00 (,75 The Decimal Parts required.

That is, $\frac{3}{4}=\frac{75}{100}=,75$.

Again $\frac{1}{2}=,5$; thus 2) 1,0(,5. And $\frac{1}{4}=,25$; 4) 1,00(,25

Suppose it were required to change $\frac{4}{7}$ into Decimals.

7) 4,0000000000 (,5714285714 &c. $=\frac{4}{7}$.

Note, When the last Figure of the Divisor, (that is, the Denominator of the proposed Fraction) happens to be one of these Figures; *viz.* 1 . 3 . 7 . or 9 . (as in the Example) then the Decimal Parts can never be precisely equal to the given Fraction; yet by continuing the Division on, you may bring them to be very near the Truth. As in this Example; Suppose it was required to Change $\frac{1}{13}$ into Decimal Parts.

```
13) 1,0000 (,0769230769230 &c. ad infinitum.
     91 ..
    ----
     90
     78
    ----
     120
     117
    -----
      39
      26
    -----
      40
      39
    -----
      10
     &c.
```

That is, 0,076923076923$07=\frac{1}{13}$ *fere.*

And from hence it may be farther observed, that in these imperfect Quotients, the Figures do return again and circulate in the same Order as before: as you may easily perceive they begin to do in the seventh Place of both these last Examples.

As at first.

These being understood, it will be easy to find the Decimal Parts equivalent to any known Part or Parts of Coin, Weights, Measures, Time, &c. If you first reduce the given Parts of Coin, &c. into a Vulgar Fraction, whose Denominator is the Number of those known Parts contained in the Integer, and the given Parts its Numerator.

Examples in Coin, &c.

1. Let it be required to find the Decimals of 16*s.* 6*d.* First 16*s.*$=\frac{16}{20}$ of one Pound, and 6*d.*$=\frac{1}{40}$ of 1*l.*

But $\frac{16}{20}+\frac{1}{40}=\frac{33}{40}$. Then 40) 33,000 (,825 the Decimal Parts required: That is, 825=16*s.* 6*d.*

Again, Suppose it were required to find the Decimals equal to 3*l.* 13*s.* 4*d.*

Here

Here 3*l*, is 3 Integers, and 13*s*.$=\frac{13}{20}$ of 1*l*. and 4*d*.$=\frac{4}{240}$. But $\frac{13}{20}+\frac{4}{240}=\frac{160}{240}$. Then 240) 160,000 (0,666666 &c. Hence 3*l*. 13*s*. 4*d*.=3,666666 &c. As was required.

2. What are the Decimals equal to $7\frac{3}{4}$ Inches, one Foot being made the Integer.

First, 7 Inches are $\frac{7}{12}$ of 1 Foot, and $\frac{3}{4}$ of 1 Inch are $\frac{3}{48}$. But $\frac{7}{12}+\frac{3}{48}=\frac{31}{48}$. Then 48) 31,000 (,64583 &c.$=7\frac{3}{4}$ Inches.

3. Let it be required to change 8 *Oz.* 19 *Pwt.* 8 *Grains* into Decimals; one Pound Troy being the Integer.

These being reduced into the least Terms, and added together, will become $\frac{4304}{5760}$ of 1 Pound.

Then 5760) 4304,000 (,74722 &c. The Decimals required.

And thus may any proposed Parts of Coin, Weights, Measures, &c. be reduced or changed into Decimal Parts; which perhaps may at first seem somewhat tedious in Practice, but being a little acquainted with them it will be found very easy; and the ingenious Practitioner will (with a little Consideration) soon find how to reduce them almost mentally; or with the Help of a very few Figures, without the Use of such large Tables as are usually inserted in Books of Decimal Arithmetick; or at most they may be contracted into such as these following, which, if duly applied to those Tables in Chap. 3. will be found very useful.

Decimal Tables.

In English Coin. 0,05 . . . . . . = 1*s*. 0,00416666 = 1*d*. 0,00104167 = 1 *Farthing*. 1*l*. being the *Integer*.	*Averdupois Weight.* 0,0625 . . . . = 1 *Ounce*. 0,00390625 = 1 *Dram*. 1 ℔. being the *Integer*.
Troy Weight. 0,05 . . . . . . = 1 *Pwt*. 0,00208333 = 1 *Grain*. 1 *Oz*. being the *Integer*.	*Averdupois Great Weight.* 0,25 . . . . . . =$\frac{1}{4}$ *C*. 0,00892857 = 1 ℔. 0,00055803 =1 *Ounce*. 1 *C*. being the *Integer*.
Apothecaries Weight. 0,125 . . . . . = 1 *Dram*. 0,04166666 = 1 *Scruple*. 0,00208333 = 1 *Grain*. 1 *Oz*. being the *Integer*.	*Time.* 0,04166666 = 1 *Hour*. 0,00069444 = 1 *Minute*. 0,00001157 = 1 *Second*. 1 *Day*, or 24 *Hours*, being made the *Integer*.

The Use of these Tables will be evident by the following

EXAMPLE.

EXAMPLE.

Let it be required to find the Decimal Parts equivalent to 17*s.* 9*d.* 2 *Farthings.*

First 0,05=1*s.* Therefore 17 × ,05=,85 .... =17*s.*
And ,004166=1*d.* Therefore ,004166×9=,037494=9*d.*
Also 2),004166(=,002083=½*d.*

Consequently their Sum, *viz.* 0,889577=17*s.* 9½*d.*

Now to find the Value of Decimals in known Parts of Coin or Weights, &c. is only the Converse of the former Work, and is thus performed.

Multiply the given Decimals with the Denominator of the Vulgar Fraction required: That is multiply the Decimals with such a Number of Units as are contained in the next lower Denomination of that Kind or Species which your Decimal is of; and the Product will be the Number required.

EXAMPLE

1. What is the Value of 0,825 Decimals of 1 Pound Sterling. That is, how many Shillings, Pence, &c.=,825. First, the next lower Denomination is 20, because 20*s.* make one Pound.

Therefore 0,825
20

Shillings, 16,500 and Parts of 1 Shilling,
12

Pence 6,000 Answer 0,825 = 16*s.* 6*d.*

Again, What are the known Parts of English Coin equal to 3,666666 Decimals.

Here the 3 Integers are 3 Pounds. Then ,666666
20

Shillings 13,333320
12

666640
3 3332

Answer 3,666666 = 3*l.* 13*s.* 4*d.*

Pence 3,999840=4 near.

What is the Value of 0,74722 Parts of 1 lb *Troy.*

First, ,74722
12

1 49444
7 4722

8,96694

Then, ,96664
20

Pwts. 19,33280

Again, ,33280
24

1 3312
6 656

7,98720

	Oz.	*Pwt.*	*Gr.*	
These collected are	8	19	8	very near.

And

And thus any proposed Number of Decimals may be turned or changed into the known Parts of what they represent. *viz.* Whether they be Parts of Coin, Weights, Measures, or Time, &c.

I have omitted inserting more Examples of this Kind, because I take the Excellency and indeed the chief Use of Decimal Fractions to consist more in Geometrical Computations than in the common or practical Parts of Arithmetick as will appear further on; although even in those they are very useful upon several Accounts; especially in the Computations of Interest and Annuities, &c. But of that more in it's proper Place. I shall therefore conclude this Chapter with a Remark or two upon the Nature and Properties of Fractions in general.

If any given Number (whether it be whole or mixed) be multiplied with a Fraction either Vulgar or Decimal, the Product will be less than the Multiplicand, in such a Proportion as the multiplying Fraction is less than an Unit or 1.

That is; as the Denominator of the Fraction is to its Numerator, so will the given Number be to the Product.

Therefore, whenever any Number is to be multiplied with a Fraction, whose Numerator is an Unit: Divide that Number by the Denominator of the Fraction, and the Quotient will be the Product required. Thus $12 \times \frac{1}{4} = 3$. And $12 \div 4 = 3$. Again, $12 \times \frac{1}{2} = 6$. And $12 \div 2 = 6$, &c.

From hence it follows, that if any Number be divided by a Fraction, the Quotient will be greater than the Dividend, by such a Proportion as Unity is greater than the dividing Fraction.

Thus $12 \div \frac{1}{4} = 48$, *viz.* $\frac{1}{4} : 1 :: 12 : 48$, &c. But the Truth of these will be best understood after the next Chapter.

CHAP. VI.

Of Continued Proportions, *and how to change or vary the Order of Things.*

Sect. 1. *Concerning* Arithmetical Progression, *usually called* Arithmetical Proportion Continued.

WHEN any Rank or Series of Numbers do either increase or decrease by an equal Interval or common Difference; those Numbers are said to be in Arithmetical Progression.

As

As { 1 . 2 . 3 . 4 . 5 . 6 . 7 &c. / 7 . 6 . 5 . 4 . 3 . 2 . 1 } { Here the Interval or common Difference is 1.

Or { 2 . 4 . 6 . 8 . 10 . 12 . 14 &c. / 1 . 3 . 5 . 7 . 9 . 11 . 13 . &c } { Here the common Difference is 2.

And so of any other Series, whose common Difference is 3 . 4 . 5 . &c.

Lemma 1.

If any three Numbers be in arithmetical Progression, the Sum of the two Extreams (*viz.* the first and last) will be equal to the Double of the Mean or middle Number.

As in these, 2 . 4 . 6 . Or 3 . 6 . 9 Or 3 . 7 . 11.

Viz. $2+6=4+4$. Or $3+9=6+6$. And $3+11=7+7$, &c.

Lemma 2.

If any four Numbers are in arithmetical Progression, the Sum of the two Extreams will be equal to the Sum of the Means.

As in these. 2 . 4 . 6 . 8 . Or 3 . 6 . 9 . 12.

Viz. $2+8=4+6$. And $3+12=6+9$, &c.

Corollary 1.

From these two Lemma's it is easy to conceive, that if never so many Numbers be in arithmetical Progression, the Sum of the two Extreams will be equal to the Sum of any two Means, that are equally distant from those Extreams.

As in these, 2 . 4 . 6 . 8 . 10 . 12 . 14 . 16.

Then $2+16=4+14=6+12=8+10$.

Or if the Number of Terms be odd, as these,

2 . 4 . 6 . 8 . 10 . 12 . 14 . 16 . 18 &c.

Then $2+18=4+16=6+14=8+12=10+10$.

Lemma 3.

Every Series of Numbers in arithmetical Progression is composed of the Interval or common Difference, so often repeated as there are Terms in the Progression, except the first.

As in these, 1 . 3 . 5 . 7 . 9 . 11 . 13 . 15 . 17, &c.

Here the Interval or common Difference being two, it will be $1+2=3$. $3+2=5$. $5+2=7$. $7+2=9$. $9+2=11$. $11+2=13$. $13+2=15$. $15+2=17$, &c.

Corollary 2.

Hence it is evident, that the Difference betwixt the two Extreams (viz. 1 *and* 17) *is compoposed of the common Difference, multiplied into the Number of all the Terms, excepting the first.*

As in the aforesaid Progression, 1. 3. 5. 7. 9. 11. 15. 17.

The Number of Terms without the first is 8 } Multiply
The common Difference is 2 }

The Difference betwixt the two Extreams 16.

Proposition 1.

In any Series of Numbers in arithmetical Progression, the two Extreams, and the Number of Terms being given, thence to find the Sum of all the Series.

Theorem. { *Multiply the Sum of the two Extreams into the Number of all the Terms; and divide the Product by 2. The Quotient will be the Sum of all that Series, Per Corol.* 1.

EXAMPLE 1.

It is required to find the Number of all the Strokes a Clock strikes in one whole Revolution of the Index, *viz.* twelve Hours.

Here $1+12=13$ the Sum of the two Extreams.
12 the Number of all the Terms.

26
13

Then 2) 156 (78. The Number of Strokes required.

EXAMPLE 2.

Suppose one Hundred Eggs were placed in a Right-Line a Yard distant from one another, and the first Egg were a Yard from a Basket; whether or no may a Man gather up these 100 Eggs singly one after another, still returning with every Egg to the Basket and putting it in, before another Man can run four Miles. That is, which will run the greater Number of Yards?

In this Question $200+2=202$ Is the Sum of the two Extr.
And 100 Is the Num. of all the Terms.

Then 2) 20200 (10100 { The Number of Yards he runs that takes up the Eggs.

Now 4 Miles=7040 Yards { The Yards he runs that takes up
But $10100-7040=3060$ { the Eggs, more than the other.

Proposition 2.

In any Series of Numbers in arithmetical Progression, the two Extreams and Number of Terms being given; thence to find the common Difference of all the Terms in that Series.

Theor. 2. { *The Difference betwixt the two Extreams, being divided by the Number of Terms less an Unit or* 1. *The Quotient will be the common Difference of the Series. Per Corol.* 2.

EXAMPLE

EXAMPLE 1.

One had Twelve Children that differed alike in all their Ages; the youngeſt was nine Years old, the eldeſt was thirty-ſix and an half; What was the Difference of their Ages, and the Age of each?

Here 36,5—9=27,5 The Difference of the two Extreams.
And 12—1=11 The Number of Terms leſs an Unit.
Then 11) 27,5 (2,5 The common Difference required.
Conſequently 9+2,5=11,5 The Age of the youngeſt but one.
And 11,5+2,5=14 The Age of the youngeſt but two. And ſo on for the reſt. *Per Corol.* 2.

EXAMPLE 2.

A Debt is to be diſcharged at eleven ſeveral Payments to be made in arithmetical Progreſſion. The firſt Payment to be Twelve Pounds Ten Shillings, and the laſt to be Sixty-three Pounds. What is the whole Debt, and what muſt each Payment be?

Per Theorem 1. Find the whole Debt thus:

12,5+63=75,5 The Sum of the Extreams.
11 The Number of Terms.

755
755

2) 830,5 (415,25=415*l.* 5*s.* The whole Debt.

Then, *per Theorem* 2. find the common Difference of each Payment.

Thus 63—12,5=50,5 The Difference of the Extreams.
And 11—1=10 The Number of Terms leſs 1.
Then 10) 50,5 (5,05=5*l.* 1*s.* The common Difference.

Conſequently 12*l.* . 10*s.*+5*l.* . 1*s.*=17*l.* . 11*s.* The ſecond Payment.

And 17*l.* . 11*s.*+5*l.* . 1*s.*=22*l.* . 12*s.* The third Payment, &c.

EXAMPLE 3.

A Man is to travel from *London* to a certain Place in ten Days, and to go but two Miles the firſt Day, increaſing every Day's Journey by an equal Exceſs; ſo that the laſt Day's Journey may be twenty-nine Miles; what will each Day's Journey be, and how many Miles is the Place he goes to diſtant from *London?*

L 2 Firſt

First 29—2=27 The Difference of the Extreams.
And 10—1=9 The Number of Terms less 1.
Then 9) 27 (3 The common Difference.
Consequently 2+3=5 The second Day's Journey.
And 5+3=8 The third Day's Journey, &c.
Again 29+2=31 The Sum of the Extreams.
10 The Number of Terms.
2) 310 (155 The Distance required.

There are eighteen Theorems more relating to Questions in arithmetical Progression; but because they would require a great many Words to shew the Reason of them: I therefore refer the Reader to the Second Part, *viz.* That of *Algebra*, where he may find their *analytical Investigation.*

Sect. 2. *Concerning* GEOMETRICAL PROPORTION *continued*; *sometimes called* GEOMETRICAL PROGRESSION.

WHEN a Rank or Series of Numbers do either increase by one common Multiplicator, or decrease by one common Divisor; Those Numbers are said to be in Geometrical Proportion continued.

As $\begin{cases} 2 . 4 . 8 . 16 . 32 . \text{\&c. here 2 is the common Multiplier.} \\ 64 . 32 . 16 . 8 . 4 . \text{\&c. here 2 is the common Divisor.} \end{cases}$

Or $\begin{cases} 2 . 6 . 18 . 54 . 162 . \text{\&c. here 3 is the common Multiplier.} \\ 162 . 54 . 18 . 6 . 2 . \text{here 3 is the common Divisor.} \end{cases}$

Note, The common Multiplier (or Divisor) is called the Ratio; and it shews the Habitude or Relation the Numbers have to one another, *viz.* whether they are Double, Triple, Quadruple, &c. which *Euclid* thus defines.

Ratio *(or* Rate) *is the mutual Habitude or Respect of two Magnitudes (consequently two Numbers) of the same Kind each to other, according to Quantity*, Euc. 5. Def. 3.

Proportion (rather Proportionality) is a Similitude of *Ratios*. Euc. 5. Def. 4.

So that there cannot be less than three Terms to form a Proportionality or Similitude of Ratio's; and if but three Terms, the second must supply the Place of two, As in these 2 . 4 . 8 . That is, 2 : 4 :: 4 : 8 . (of :: see *page* 5)

Here 4 the middle Term supplies the Place of two Terms, to wit, of the second and third; 8 bearing the same Reason, Likeness,

Likenefs, or Proportion to 4, As 4 doth to 2, *viz.* As 2 : is to 4 : : So is 4 : to 8.

Lemma 1.

If three Numbers are proportional, the Rectangle or Product of the two Extreams, *viz.* of the firft and laft Terms, will be equal to the Square of the mean or middle Term. (20 *Eucl.* 7.)

As in thefe 2 : 4 : : 4 : 8. Here $8\times2=16$ the Product of the Extreams.

And $4\times4=16$ the Square of the Mean. *Ergo* $8\times2=4\times4$.

Corol. 1.

Hence it follows, that if the Product of any two Numbers be equal to the Square of a third Number; thofe three Numbers will be in proportion.

Lemma 2.

If four Numbers are proportional, the Product of the two Extreams will be equal to the Product of the two Means (19 *Euclid* 7.)

As in thefe, 2 : 4 : : 8 : 16. Here $16\times2=32$.
And $8\times4=32$. Confequently $16\times2=8\times4$.

Corol. 2.

From Hence it follows, that if the Product of any two Numbers, be equal to the Product of any other two Numbers, thofe four Numbers are Proportionals.

And from thefe two *Lemma*'s it will be eafy to conceive, that if never fo many Numbers are in continued Proportion; the Product of the two Extreams, will be equal to the Product of any two Means, that are equally diftant from the Extreams.

As in thefe 2 . 4 . 8 . 16 . 32 . 64 . &c.

Here $64\times2=32\times4=16\times8$, &c. And if the Number of Terms be odd,

As in thefe 2 . 4 . 8 . 16 . 32 . 64 . 128 . &c.
Then $128\times2=64\times4=32\times8=16\times16$.

Note, *The Character made ufe of to fignify continued Proportionals is* $\div\!\!:$,

In

In every Series of ∺ *(viz. of continued Proportionals)* that Number which is compared to another, is called the Antecedent of the Ratio; and that Number to which it is compared, is called its Consequent.

As in these, 2 : 4 :: 4 : 8. Here 2 is the Antecedent, and 4 is the Consequent; and 4 the middle Term is an Antecedent to 8 its Consequent: whence it follows, that in every Series of ∺ all the middle Terms between the first and last are both Antecedents and Consequents.

As in these, 2 . 4 . 8 . 16 . 32 . 64 *&c.* Here 4 . 8 . 16 . 32. are both Consequents and Antecedents.

For 2 : 4 :: 4 : 8 :: 8 : 16 :: 16 : 32 :: 32 : 64 *&c.*

So that all the Terms except the last are Antecedents. And all the Terms except the first are Consequents.

Lemma 3.

If never so many Numbers are proportional, it will be: As any one of the Antecedents is to its Consequent; So will the Sum of all the Antecedents be, to the Sum of all the Consequents. (12 *Euclid* 5.)

That is, in the foregoing Series.

$2 : 4 :: 2+4+8+16+32 : 4+8+16+32+64.$

For it is evident, that $4+8+16+32+64$ the Sum of all the Consequents, is double to $2+4+8+16+32$ the Sum of all the Antecedents; as 4 is to 2, according to the Ratio, and would have been Triple, or Quadruple, *&c.* had the Ratio been 3 or 4, *&c.*

Note, *In every Series of ∺ the Ratio is found by dividing any of the Consequents by its Antecedent.*

As in these 2 : 6 :: 6 : 18 :: 18 : 54 :: 54 : 162.

Here 2) 6 (3 the Ratio. Or 6) 18 (3 *&c.*

From the second and third Lemma's may be raised two general Theorems or Rules, for finding the Sum of any Series in ∺ without a continued Addition of all the Terms.

Let the Series 2 . 4 . 8 . 16 . 32 . 64 . 128. be given, to find its Sum.

Suppose $z=$ the Sum of all the Terms.
Then will $z-128=$ the Sum of all the Antecedents.
And $z-2=$ the Sum of all the Consequents.
But $2 : 4 :: z-128 : z-2$. *per Lemma* 3.
Ergo $4z-512=2z-4$. *per Lemma* 2.

Consequently

Consequently $4z-2z=512-4$.

Theorem. $\left\{ z=\frac{512-4}{4-2} \right.$ In Words at length thus,

Theorem 1. $\left\{ \right.$ *From the Product of the second and last Terms substract the Square of the first Term, and that Remainder being divided by the second Term less the first, will give the Sum of all the Series.*

Or if the first Term, the common Ratio, and the last Term be only given. Then,

Theorem 2. $\left\{ \right.$ *Multiply the last Term into the Ratio, and from their Product substract the first Term; divide that Remainder by the Ratio less Unity or* 1, *and it will give the Sum of all the Series.*

For $4z-2z=512-4$. As above.
Consequently $2z-z=256-2$ *viz.* the last divided by 2.

Then $z=\frac{256-2}{2-1.}$ *Theorem* 2.

EXAMPLE.

Let 2 . 6 . 18 . 54 . 162 . 486. be the given Series. Here 2 is the first Term, 3 is the Ratio, and 486 the last Term.

But $486\times3=1458$. And $1458-2=1456$.
Then $3-1=2$) 1456 (728 the Sum required.
That is, $728=2+6+18+54+162+486$.

Since in either of these Theorems it is required to have the last Term known (the which in a long Series of ∺ will be very tedious to come at by a continued Multiplication; it will therefore be convenient to shew how to obtain either the last Term or any other Term, whose Place is assigned, without producing all the Terms.

In order to that, it will be necessary to premise the Coherence or Similitude that is betwixt Numbers in arithmetical Progression and those in geometrical Proportion.

If to any Series of Numbers in ∺ when the first Term is not an Unit or 1, there be assigned a Series of Numbers in arithmetical Progression, beginning with an Unit or 1, and whose common Difference is 1. called Indices or Exponents.

Thus, $\left\{ \begin{array}{l} 1 . 2 . 3 . 4 . 5 . 6 . 7 \text{ Indices} \\ 2 . 4 . 8 . 16 . 32 . 64 . 128 \text{ \&c. ∺} \end{array} \right.$

Then

Then will the Addition or Substraction of any two of those Indices (or Numbers in arithmetical Progression) directly correspond with the Product, or Quotient, of their respective Terms in the Series of ∺.

That is, $\begin{cases} \text{As } 3+4=7 \\ \text{So } 8\times16=128 \text{ the seventh Term in } ∺ \end{cases}$

Again, $\begin{cases} \text{As } 6+4=10 \\ \text{So } 64\times16=1024 \text{ the tenth Term in } ∺ \end{cases}$

Or, $\begin{Bmatrix} \text{As } 7-3=4 \\ \text{So } 128 ∺ 8=16 \end{Bmatrix}$ Or, $\begin{Bmatrix} \text{As } 6-2=4 \\ \text{So } 64 ∺ 4=16 \end{Bmatrix}$ *&c.*

But if the Series of ∺ begin with an Unit, the Indices must begin with a Cypher.

As in these, $\begin{cases} 0 \,.\, 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6, \text{ \&c.} \\ 1 \,.\, 2 \,.\, 4 \,.\, 8 \,.\, 16 \,.\, 32 \,.\, 64. \end{cases}$

Now by the Help of the Indices, and a few of the first Terms in any Series of ∺, it is plain, that any Term whose Place or Distance from the first Term is assigned, may be speedily obtained without producing the whole Series.

EXAMPLE 1.

A Man bought a Horse, and was to give a Farthing for the first Nail, two for the second, four for the third, *&c.* in ∺, the Number of Nails was to be 7 in every Shoe, *viz.* 28 Nails in all. What must he have paid for the Horse?

First $\begin{cases} 0 \,.\, 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \text{ Indices} \\ 1 \,.\, 2 \,.\, 4 \,.\, 8 \,.\, 16 \,.\, 32 \text{ Farthings in } ∺ \end{cases}$

Then, $\begin{cases} 5+5=10 \\ 32\times32=1024 \end{cases}$ And $\begin{cases} 10+10=20 \\ 1024\times1024=1048576 \end{cases}$

Again, $\begin{cases} 4+3=7 \\ 16\times8=128 \end{cases}$ Lastly, $\begin{cases} 20+7=27 \\ 1048576\times128=134217728 \end{cases}$

Which is here to be accounted the 28 and last Term. Because the first Term in the Series is 1, which doth neither multiply nor divide.

Now this 134217728 being the Number of Farthings to be paid for the last Nail, by it the common Ratio which is 2, and the first Term which is 1, may be found the Sum of all the Series, *per Theorem* 2.

$$\begin{array}{r} 134217728 \\ 2 \\ \hline 268435456 \end{array}$$

268435456 From this Product substract 1.

Viz. 268435456—1=268435455. Then 2—1=1 the Divisor.

Consequently 268435455 is the Sum of all the Series, or Price of the Horse in Farthings, which being brought into Pounds, (See *page* 46) will be 279620*l.* 5*s.* 3*d.* 3*qrs.*

EXAMPLE 2.

A cunning Servant agreed with a Master (unskilled in Numbers) to serve him eleven Years without any other Reward for his Service but the Produce of one Wheat Corn for the first Year, and that Product to be sowed the second Year, and so on from Year to Year until the End of the Time, allowing the Increase to be but in a ten-fold Proportion.

It is required to find the Sum of the whole Produce.

First $\begin{cases} 1 \;.\; 2 \;.\; 3 \;.\; 4 \;.\; 5 \;. & \text{Indices or Years.} \\ 10 \;.\; 100 \;.\; 1000 \;.\; 10000 \;.\; 100000 & \text{Wheat Corns in } ∺ \end{cases}$

Then $\begin{cases} \text{As } 4+2=6 \\ \text{So } 10000\times100=1000000 \text{ the 6th Year's Produce.} \end{cases}$

And $\begin{cases} 6+5=11 \\ 1000000\times100000=100000000000 \text{ the Eleventh or last} \end{cases}$
Year's Produce.

Then (either by *Theorem* 1, or 2) the Sum of all the Series will be 111111111110 Corns. Now it may be computed from Page 31 and 34, that 7680 Wheat Corns, round and dry out of the Middle of the Ear, will fill a Statute Pint. If so,

Then 7680) 11111111110 (14467592 Pints, but 64 Pints are contained in a Bushel.

Therefore 64) 14467592 (226056$\frac{1}{8}$ Bushels. Suppose it to be sold for 3 Shillings the Bushel;

Then $\begin{cases} 226056\frac{1}{8} \\ 3 \end{cases}$

Shillings 678168$\frac{3}{8}$=33908*l.* 8*s.* 4$\frac{1}{2}$*d.* A very good Recompence for eleven Years Service.

There are several pretty Questions resolved by Numbers in arithmetical Progression; and by those in ∺ which the ingenious Learner will easily perceive hereafter, *viz.* when we come to the Solution of Questions relating to Interest and Annuities, &c.

There is alſo a third Kind of Proportion, called Muſical, which being but of little or no common Uſe, I ſhall therefore give but a ſhort Account of it.

Muſical Proportion or Habitude is, when of three Numbers; the firſt hath the ſame Proportion to the third, as the Difference between the firſt and ſecond hath to the Difference between the ſecond and third.

As in theſe, 6 . 8 . 12 *viz.* 6 : 12 : : 8—6 : 12—8

If there are four Numbers in muſical Proportion; the firſt will have the ſame Proportion to the fourth, as the Difference between the firſt and ſecond hath to the Difference between the third and fourth.

As in theſe 8 . 14 . 21 . 84.

Here 8 : 84 : : 14—8=6 : 84—21=63.

That is, 8 : 84 : : 6 : 63.

The Method of finding out Numbers in muſical Proportion, is beſt expreſſed by Letters; as ſhall be ſhewed in the *algebraic Part.*

Sect. 3. *How to* Change *or* Vary *the Order of Things,* &c.

THIS being a Thing not treated of in any common Books of Arithmetick (that I have had the Opportunity of peruſing) made me think it would be acceptable to the young Learner, to know how oft it is poſſible to vary or change the Order or Poſition of any propoſed Number of Things.

As how many ſeveral Changes may be rung upon any propoſed Number of Bells; or how many ſeveral Variations may be made of any determined Number of Letters, or any other Things propoſed to be varied.

The Method of finding out the Number of Changes is by a continued Multiplication of all the Terms in a Series of arithmetical Progreſſions, whoſe firſt Term and common Difference is Unity or 1. *And the laſt Term the Number of Things propoſed to be varied,* viz. 1×2×3×4×5×6×7, *&c.* As will appear from what follows.

1. If the Things propoſed to be varied are only two, they admit of a double Poſition (as Order of Place) and no more.

$$\text{Thus, } \left\{ \begin{matrix} 1 . 2 \\ 2 . 1 \end{matrix} \right\} = 2 - 1 \times 2$$

2. And if three Things are propoſed to be varied, they may be

be changed ſix ſeveral Ways, as to their Order of Place, and no more.

For, beginning with 1, there will be $\begin{cases} 1 . 2 . 3 \\ 1 . 3 . 2 \end{cases}$

Next, beginning with 2, there will be $\begin{cases} 2 . 1 . 3 \\ 2 . 3 . 1 \end{cases}$

Again, beginning with 3 it will be $\begin{cases} 3 . 1 . 2 \\ 3 . 2 . 1 \end{cases}$

Which in all make 6 or 3 Times 2, *viz.* $1\times2\times3=6$

Suppoſe four Things are propoſed to be varied; Then they will admit of 24 ſeveral Changes, as to their Order of different Places.

For beginning the Order with 1 it will be $\begin{cases} 1 . 2 . 3 . 4 \\ 1 . 2 . 4 . 3 \\ 1 . 3 . 2 . 4 \\ 1 . 3 . 4 . 2 \\ 1 . 4 . 2 . 3 \\ 1 . 4 . 3 . 2 \end{cases}$

And for the ſame Reaſon there will be 6 different Changes, when 2 begins the Order, and as many when 3 and 4 begins the Order; which in all is $24=1\times2\times3\times4$. And by this Method of proceeding, it may be made evident, that 5 Things admit of 120 ſeveral Variations or Changes; and 6 Things of 720, &c. As in this following Table.

The Number of Things propoſed to be varied.	*The Manner how their ſeveral Variations are produced.*	*The different Changes or Variations every one of the propoſed Numbers can admit of*
1	1	=1
2	1×2	=2
3	2×3	=6
4	6×4	=24
5	24×5	=120
6	120×6	=720
7	720×7	=5040
8	5040×8	=40320
9	40320×9	=362880
10	362880×10	=3628800
11	3628800×11	=39916800
12	39916800×12	=479001600
&c.	&c.	&c.

 These

These may be thus continued on to any assigned Number. Suppose to 24 the Number of Letters in the Alphabet, which will admit of 620448401733239439360000 several Variations.

From these Computations may be started several pretty, and indeed, very strange Questions.

EXAMPLES.

Six Gentlemen, that were travelling, met together by Chance at a certain Inn upon the Road, where they were so pleased with their Host, and each other's Company, that in a Frolick they made a Contract to stay at that Place, so long as they, together with their Host, could sit every Day in a different Order or Position at Dinner; which by the foregoing Computations will be found near 14 Years. For they being made 7 with their Host, will admit of 5040 different Positions; but 5040 being divided by $365\frac{1}{4}$ (the Number of the Days in one Year) will give 13 Years and 291 Days. A very pretty Frolick indeed.

I have been told, that before the Fire of *London* (which happened *Anno* 1666) there were 12 Bells in St. *Mary Le Bow*'s Church in *Cheapside*, *London*. Suppose it were required to tell how many several Changes might have been rung upon those 12 Bells; and at a moderate Computation, how long all those Changes would have been ringing but once over.

First, $1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11\times12=479001600$ the Number of Changes.

Then supposing there might be rung 10 Changes in one Minute, *viz.* $12\times10=120$ Strokes in a Minute, which is 2 Strokes in a Second of Time: Now according to that Rate, there must be allowed 47900160 Minutes to ring them once over in all their different Changes; *viz.* 10) 479001600 (47900160.

In one Year there is 365 Days, 5 Hours, and 49 Minutes; which, being reduced into Minutes, is 525949.

Then 525949) 47900160 (91 Years, and 26 Days.

So long would those 12 Bells have been continually ringing without any Intermission, before all their different Changes could have been truly rung but once over. It is strange, and seems almost incredible, that a few Things should produce such Varieties.

But that which seems yet more strange and surprising (yea, even impossible to those who are not versed in the Power of Numbers) is,

is, that if two Bells more had been added to the aforesaid 12 they would have advanced the Number of Changes, and consequently the Time, beyond common Belief. For 14 Bells would require (at the same Rate of ringing as before) about 16575 Years to ring all their different Changes but once over.

And if it were possible to ring 24 Bells in Changes (and at the same Rate of 10 Changes in a Minute, which is 2 Strokes in one Second) they would require more than 117000000000000000 Years to ring them but once over in all their different Changes; as may easily be computed from the precedent Table.

CHAP. VII.

Of Proportion Disjunct; *commonly called the* Golden Rule.

Proportion Disjunct, or the Golden Rule, is either *Direct* or *Reciprocal*, called *Inverse*. And those are both Simple and Compound.

SECT I.

Direct Proportion is, when of four Numbers, the first beareth the same Ratio or Proportion to the second; as the third doth to the fourth.

As in these $2 : 8 :: 6 : 24$.

Consequently, the greater the second Term is, in respect to the first; the greater will the fourth Term be, in respect to the third.

That is, as 8 the second Term, is 4 Times greater than 2 the first Term: So is 24 the fourth Term, 4 Times greater than 6 the third Term.

Whence it follows, that if four Numbers are in direct Proportion, the Product of the two Extreams will always be equal to the Product of the two Means, as well in Disjunct as in continued Proportion; according to *Lemma* 2. *page* 77.

For as $2 : 2\times4 :: 6 : 6\times4$. Or as $3 : 3\times5 :: 6 : 6\times5$.
But $2\times6\times4=2\times4\times6$. Or $3\times6\times5=3\times5\times6$.

That is, the Product of the Extreams is equal to that of the Means.

Again,

Again, the less the second Term is, in respect to the first; the less will the fourth Term be in respect to the third.

As in these $18 : 6 :: 12 : 4$.
That is, $18 : 18 \div 3 :: 12 : 12 \div 3$.

But $18 \times 12 \div 3 = 18 \div 3 \times 12$. *Viz.* $18 \times 4 = 6 \times 12$.
Consequently $2 . 8 . 6 . 24$. And $18 . 6 . 12 . 4$. are true Proportionals, *per Corol. 2. page* 77.

From these Considerations, comes the Invention of finding a fourth Number in Proportion to any three given Numbers. *Whence it is called the Rule of Three.*

For if the second Number multiplied into the third, be equal to the first multiplied into the fourth, it is easy to conceive, that if the Product of the second and third be divided by the first, the Quotient must needs be the fourth Number. For if that Number, which divides another, be multiplied into the Quotient produced by that Division, their Product will be equal to the Number divided. *See page* 21.

As in these $2 : 8 :: 6 : 24$. Here $8 \times 6 = 48 = 24 \times 2$.
But if $24 \times 2 = 48$, then will $48 \div 2 = 24$. Or $48 \div 24 = 2$.

Note, Any four Numbers in direct Proportion may be varied several Ways. As in these.

Viz. If $2 : 8 :: 6 : 24$. Then $2 : 6 :: 8 : 24$.
And $6 : 24 :: 2 : 8$. Or $24 : 6 :: 8 : 2$. &c.

These Variations being well understood, will be of no small Use in the stating of any Question in this Rule of Three.

When three Numbers are given, and it is required to find a fourth Proportional; the greatest Difficulty (if there be any) will be in the right stating the Question, or abstracting the Numbers out of the Words in the Question, and placing them down in their proper Order.

Now this will be very easy, if it be truly considered, that *always two* of the three given Terms, are *only supposed*, and assign or limit the Ratio or Proportion. The *third moves* the Question; and the fourth gives the Answer.

As for Instance, if 3 Yards of Cloth cost 9 Shillings: What will 6 Yards cost at the same Rate or Proportion?

Here 3 Yards, and 9 Shillings, are two supposed Numbers that imply the Rate; as appears by the Word [if] *viz.* If 3 Yards cost 9 Shillings (then comes the Question) What will 6 Yards cost?

N. B.

N. B. The Term, which moves the Queſtion, hath generally ſome of thoſe Words before it; *viz.* WHAT WILL? HOW MANY? HOW LONG? HOW FAR? or HOW MUCH? &c.

Then *carefully obſerve this,* viz.) The *firſt Term* in the Suppoſition muſt always be of the ſame Kind and Denomination with *that Term* which moves the Queſtion. And the Term ſought will always be of the ſame Kind and Denomination with the ſecond Term in the Suppoſition.

Thus { *yds ſhil. yds ſhil.*
3 : 9 :: 6 : —— Then

All Queſtions in direct Proportion may be anſwered by three ſeveral *Theorems.*

Theorem 1. { *Multiply the ſecond and third Term together, and divide their Product by the firſt Term; the Quotient will be the Anſwer required.*

yds ſhil. yds ſhil.
Thus 3 : 9 :: 6 : 18. The Anſwer.
6

3) 54 (18 Shillings, { becauſe the ſecond Term was Shillings.

Theorem 2. { *Divide the ſecond Term by the firſt, then multiply the Quotient into the third Term; and their Product will be the Anſwer required.*

yds ſhil. yds ſhil.
3 : 9 :: 6 : 18.
Thus 3) 9 (=3. Then 3×6=18, as before.

Theorem 3. { *Divide the third Term by the firſt, then multiply the Quotient into the ſecond Term, and their Product will be the Anſwer.*

yds ſhil. yds ſhil.
3 : 9 :: 6 : 18.
Thus 3) 6 (=2. And 9×2=18, as before.

Here you ſee that all the three *Theorems* are equally true; *but* the *firſt* is moſt *general,* and *uſually practiſed.* Yet the two laſt may be readily performed, when either the ſecond or third Term can be divided by the firſt; and will be found of ſingular Uſe in the *Rules of Fellowſhip,* &c. as will appear further on.

Queſt.

Queſt. 2. If 8 Pounds of Tobacco coſt 14 *Shillings*; what will half a hundred Weight (*viz*. 56 Pounds) coſt at the ſame Rate?

Thus 8 ℔ 14*s*. : : 56 ℔ : 4*l*. 18*s*. The Anſwer.

$$\begin{array}{r} 14 \\ \hline 224 \\ 56 \\ \hline \end{array}$$

8) 784 (=98*s*.=4*l*. 18*s*.

Or thus 8) 56 (=7 Then 14×7=98*s*. as before.

Queſt. 3. If 14 *Shillings* will buy 8 Pounds of *Tobacco*; how much will 4*l*. 18*s*. buy after the ſame Rate?

Stated thus, 14*s*. : 8 lb. : : 4*l*. 18*s*.=98*s*. : ———
Then 98×8=784. And 14) 784 (56 lb. The Anſwer.

Queſt. 4. If half a hundred Weight of *Tobacco* be worth 4*l*. 18*s*. How much may I buy for 14 *Shillings* at that Rate?

Stated thus, 4*l*. 18*s*=98*s*. : 56 lb. : : 14*s*. : ———
Then 56×14=784. And 98) 784 (8 lb. The Anſwer.

Queſt. 5. Suppoſe 4*l*. 18*s*. will buy 56 Pounds of *Tobacco*; what will 8 Pounds of the ſame *Tobacco* coſt?

This Queſtion is thus ſtated. 56 lb. : 4*l*. 18*s*.=98*s*. : : 8 lb. : ———
Then 98×8=784. And 56) 784 (=14*s*. The Anſwer.

Note, The three laſt Queſtions are only the ſecond varied, being propoſed purely to give an Inſtance how any Queſtion in this *Rule of Three* may be varied, according to *page* 86.

Queſt. 6. What will three quarters of a Yard of *Velvet* coſt, when the Price of 21 Yards and a half is worth 22*l*. 10*s*. 6*d*. This Queſtion truly ſtated will ſtand

Thus, 21 $\frac{1}{2}$ *yards* : 22*l*. 10*s*. 6*d*. : : $\frac{3}{4}$ To the Anſwer.

Which may be found three ſeveral Ways, *viz*. by *Reduction*; by *Vulgar Fractions*; and by *Decimals*.

1. By *Reduction*. Bring the firſt and third Terms into one Denomination, *viz*. into Quarters, and reduce the ſecond Term into its leaſt Denomination, *per Sect* 4. *page* 42.

Thus 21$\frac{1}{2}$=86 Quarters. And 22*l*. 10*s*. 6*d*.=5406 *Pence*.
Then 86 ; 5406 : : 3 : 15*s*. 8$\frac{50}{86}$*d*. For 5406×3=16218.

And

And 86) 16218 (=188$\frac{50}{86}$*d.* Then 188$\frac{50}{86}$ *Pence* = 15*s.* 8*d.* 2$\frac{14}{43}$ *Farthings*; the Anſwer required.

2. The ſame Queſtion ſtated in *Vulgar Fractions* will ſtand Thus $21\frac{1}{2}=\frac{43}{2}$: $22\frac{21}{40}=\frac{901}{40}$ :: $\frac{3}{4}$: (See *Sect.* 3. *page* 50.) Then $\frac{901}{40}\times\frac{3}{4}=\frac{2703}{160}$. And $\frac{43}{2}$)$\frac{2703}{160}$(= $\frac{5406}{6880}$ *page* 55, 56.

These $\frac{5406}{6880}$ Parts of a Pound are brought into *Shillings* by Multiplying the Numerator with 20, and dividing the Product by its Denominator, &c.

Thus 5406×20= 108120. And 6880) 108120 (15*s.*
And there remains 4920. Again 4920×12=59040.
Then 6880) 59040 (8*d.* and $\frac{50}{86}$, as before.

3. The ſame wrought by *Decimal Fractions* will be thus;
$21\frac{1}{2}$=21,5; 22*l.* 10*s.* 6*d.*=22,525, and $\frac{3}{4}$=0,75
Therefore 21,5 : 22,525 :: 0,75 : to the Anſwer.
Then 22,525×0,75= 16,89375
And 21,5) 16,89375 (0,7857*l.* = 15*s.* 8*d.* 2 *far.* $\frac{272}{1000}$

Queſt. 7. If 2*C.* 3*qrs.* 21 ℔. of *Sugar* coſt 6*l.* 1*s.* 8*d.* What will 12*C.* 2*qrs.* coſt at the ſame Rate?

That is, 2*C.* 3*qrs.* 21 ℔ : 6*l.* 1*s.* 8*d.* :: 12*C.* 2*qrs.* To what?

4	20	4
11 *qrs.*	121 *s.*	50 *qrs.*
28	12	28
88	250	1400℔.
22	121	

Viz. 308+21=324℔ : 1460*d.* :: 1400℔ : ——

Then 1460×1400=2044000. And 329) 2044000 (6212$\frac{3}{4}$*d.* =25*l.* 17*s.* 8$\frac{3}{4}$*d.* the Anſwer required.

The ſame Queſtion ſtated in Decimals will ſtand
Thus 2,9375 : 6,0833 :: 12,5 : To the Anſwer.

Then 6,0833×12,5=76,04125 which being divided by 2,9375 will give 25,8863, &c. the Anſwer in Decimals, which brought into Coin, will be 25*l.* 17*s.* 8$\frac{3}{4}$*d.* as before.

Note, *When the firſt Term is an Unit or* 1, *the Queſtion is anſwered by Multiplication only.*

Example. Suppoſe I give 5 *Shillings* 4 *Pence* for one Ounce of Silver, What muſt I pay for 32$\frac{1}{2}$ Ounces at the ſame Rate?

That is 1 Ounce : 5*s.* 4*d.* :: 32$\frac{1}{2}$ Ounces : To, &c.
Which is beſt ſtated thus 1 : 64*d.* :: 32,5 :

 Then

Then $32,5 \times 64 = 2080d. = 8l.\ 13s.\ 4d.$ the Anſwer required. For 1 neither multiplies nor divides.

When the ſecond or third Term is an Unit or 1, then the Queſtion is anſwered by Diviſion only. As in this *Example*.

If a Silver Tankard weighing 21 Ounces, coſt 5*l.* 19*s.* What is that an Ounce?

Thus 21 *oz.* : 5*l.* 19*s.* = 119*s.* :: 1 : 5*s.* 8*d.* the Anſwer.
That is 21) 119 (=5*s.* $\frac{14}{21}$=5*s.* 8*d.*

The Proof of all Queſtions in the *Rule of Three Direct*, may be eaſily conceived from what hath been already ſaid; *viz.* That the Product of the firſt and fourth Terms, muſt always be equal to the Product of the ſecond and third Terms.

Or otherwiſe, by varying the Queſtion, as in the ſecond, third, fourth, and fifth Queſtions.

I ſhall conclude this Section with inſerting a few Queſtions and their Anſwers; leaving their Work for the Learner's Practice.

Queſt. 1. What will the Carriage of 17*C.* 3*qrs.* 11℔. come to, at the Rate of 7*s.* the Hundred?

Anſwer 6*l.* 4*s.* 11$\frac{1}{4}$*d.*

Queſt. 2. If 6*l.* 4*s.* 11$\frac{1}{4}$*d.* be paid for the Carriage of 17*C.* 3*qrs.* 11℔.; What was paid for the Carriage of 1℔?

Anſwer 3 Farthings.

Queſt. 3. A *Grocer* bought 3 *C.* 1 *qr.* 14℔, Weight of *Cloves*, at the Rate of 2*s.* 4*d.* *per Pound*, and ſold them for 52*l.* 14*s.* Whether did he gain or loſe by the Bargain, and how much?

Anſwer he gained 8*l.* 12*s.*

Queſt. 4. A *Draper* bought of a Merchant eight Packs of *Cloth*; every Pack had four Parcels in it; and each Parcel contained ten Pieces; every Piece was twenty-ſix Yards; he gave after the Rate of four Pounds ſixteen Shillings for 6 Yards. What came the eight Packs to, and what were they worth *per Yard?*

Anſw. They came to 6656*l.* And were worth 16*s.* *per Yard.*

Queſt. 5. A Merchant bought 436 Yards of Broad *Cloth* for 8*s.* 6*d.* *per Yard*; and ſold it again for 10*s.* 4*d.* *per Yard*. What did he gain by the 436 *Yards?*

Anſwer, he gained 39*l.* 19*s.* 4*d.*

Queſt.

Queſt. 6. A *Goldſmith* bought a Wedge of *Gold*, which weighed 14lb. 3*oz.* 8*pw.* for 514*l.* 4*s.* What did he pay *per Ounce?*

Anſw. 3*l. per Ounce.*

Queſt 7. What will 48*oz.* 17*pw.* 20 *Grains of Silver* Plate come to, at the Rate of 5*s.* 6*d. per Ounce?*

Anſw. 13*l.* 8*s.* $10\frac{3}{4}$*d.*

Queſt. 8. If in four Weeks one ſpend 13*s.* 4*d.* How long will 53*l.* 6*s.* laſt at that Rate?

Anſw. 6 Years, 47 Days, 2 Hours $24^{\prime}$.

Queſt. 9. What will the one-eigth Part of a Ship be worth, when the half is valued at 1015*l.* 10*s.*

Anſw. 253*l.* 17*s.* 6*d.*

Queſt. 10. The Sun is ſaid to perform one entire Revolution, (or 360 Degrees) in the Space of 365 Days, 5 Hours, 48 Minutes, and 57 Seconds of Time, called a *Tropical* or *Solar Year*; How much doth it move in one Day?

Anſw. $59^{\prime}.\,8^{\prime\prime}.\,19^{\prime\prime\prime}$ &c.

Queſt. 11. If $\frac{5}{8}$ of a Yard of *Velvet* coſt $\frac{2}{3}$ of a *Pound Sterling*, What will $\frac{1}{16}$ of a Yard coſt of the ſame *Velvet* at that Rate?

Anſw. $\frac{16}{240}$=1*s.* 4*d.*

Queſt. 12. Suppoſe 2*l.* and $\frac{3}{8}$ of $\frac{1}{3}$ of a *Pound Sterling* will buy 3 Yards and $\frac{2}{3}$ of $\frac{3}{5}$ of a Yard of *Cloth*, How much will $\frac{3}{4}$ of a Yard coſt at that Rate?

Anſw. $\frac{2295}{4896}$ of a Pound =9*s.* $4\frac{1}{2}$*d.*

Sect. 2. *Of* Reciprocal Proportion; *uſually called* The Rule *of* Three Inverſe.

Reciprocal Proportion is, when of four Numbers the third (*viz.* that which moves the Queſtion) beareth the ſame Ratio to the firſt: As the ſecond does to the fourth.

Therefore, the leſs the third Term is, in reſpect to the firſt; the greater will the fourth Term be, in reſpect to the ſecond.

EXAMPLE 1.

If ſixteen Men can do a Piece of Work in ſix Days; How many Days muſt eight Men require to do the ſame Work, at the ſame Rate of working?

Here it is plain that eight Men muſt needs have more Time than 16 Men to do the ſame Work. Conſequently the greater

 the

the third Term is, in respect to the first, the lesser will the fourth Term be, in respect to the second.

Example 2. If 8 Men can do a Piece of Work in 12 Days, How many Days will 16 Men require do the same Work? Here it is plain the fourth Term must be less than the second, because 16 Men undoubtedly can do the Work in less Time than 8 Men can.

From these Considerations, compared with those in page 85. *it will be easy to perceive, whether the Terms of any proposed Question are in Direct or Reciprocal Proportion.*

For when, according to the true Meaning and Design of any Question in Proportion, More requires More, or Less requires Less, the Terms are in Direct Proportion; as in this last Section.

But if More require Less, or Less require More (as above) then the Terms will be in Reciprocal Proportion.

The Manner of placing down the proposed Terms is the same in both Rules, *viz.* The first Term in the Supposition must be of the same Kind and Denomination with the third Term which moves the Question; and the Term sought must be of the same Kind and Denomination with the second Term in the Supposition. As in the two last *Examples*.

	Men		Days		Men		Days
Thus, in *Example* 1.	16	:	6	: :	8	:	——
Thus, in *Example* 2.	8	:	12	: :	16	:	——

The Question being truly stated, observe this *Theorem.*

Theorem. { *Multiply the first and second Terms together, and divide the Product by the third Term, the Quotient will be the Answer required.*

Thus in the second *Example* $12 \times 8 = 96$.
Then 16) 96 (=6 Days the Answer required.

That is, 16 Men may do the same Work in 6 Days as 8 Men can do in 12 Days.

Now the Reason of this Operation (and consequently of the *Theorem*) is grounded upon this Consideration; *viz* If 8 Men require 12 Days to do the Work, it is plain that one Man would require 8 Times 12 Days=96 Days to do the same Work; but if one Man can do it in 96 Days, most certain 16 Men can do it in one 16th Part of that Time. Therefore 96 divided by 16 will give the Answer required, *viz.* 16) 96 (6 as before, &c.

Quest. 3. Suppose 800 Soldiers were besieged in a Town, and their Victuals were computed to serve them two Months (or 56 Days) How many of those Soldiers must depart the Garrison, that the same Victuals may serve the remaining Soldiers 5 Months?

The

The Queſtion truly ſtated will ſtand

Months. Soldiers. *Months. Soldiers.*

Thus, 2 : 800 : : 5 : ——
2

5) 1600 (320 : So many Soldiers may ſtay in the Garriſon.

Conſequently, 800—320=480 Soldiers that muſt go out of the Garriſon, which is the Anſwer required.

Queſtion 4. *A* borrowed of his Friend *B* 250*l.* for ſix Months, promiſing to do him the like Kindneſs upon Demand: Some Time after *B* deſires *A* to lend him 400*l.* the Queſtion is, how long *B* muſt keep the 400*l.* to be fully ſatisfied for his former Kindneſs to *A*?

Thus 250*l.* : 6 Months : : 400*l.* : ——
6

400) 1500 (3 Months.
12

3
28 Days in one Month,

4) 84 (21 Days. Anſw. 3 Months, 21 Days.

Queſtion 5. If a Penny White Loaf ought to weigh eight Ounces *Troy Weight*, when Wheat is ſold for ſix Shillings Six-Pence the Buſhel, what muſt it weigh when Wheat is ſold for four Shillings the Buſhel?

Thus 6*s.* 6*d.*=78*d.* : 8*oz.* : : 4*s.*=48*d.* : to the Anſwer.
8

48) 624 (13 *oz.* the Anſwer required.
48

144
144

(0)

The Proof of this Inverſe Rule is eaſily deduced from its Operations; *viz.* The Product of the firſt and ſecond Terms, muſt be equal to the Product of the third and fourth Terms.

Note, Any Queſtion that falls under this Inverſe Rule or Reciprocal Proportion, may be ſo ſtated as to have its Terms in Direct Proportion; by only changing the Places of the firſt and third Terms in the Queſtion. Thus,

Queſtion.

Question 6. If a Field will feed eighteen Horses for seven Weeks: how long will it feed Forty-two Horses at the same Rate of feeding?

First, 18 Horses : 7 Weeks :: 42 Horses : 3 Weeks.

Here the Terms are stated inversely, as before.

Otherwise thus, 42 Horses : 7 Weeks :: 18 Horses : 3 Weeks. Then $18\times7=126$. And $126\div42=3$ Weeks. The Answer required.

Sect. 3. *Of* Compound Proportion; *commonly called* The Double Rule of Three.

Compound Proportion (as it is here meant) is when there are five Numbers given to find out a sixth Proportional; and this is generally performed by a *Double Position*; that is, by stating and working the Question at *two Operations*, either in Direct or Reciprocal Proportion, according as the Question requires.

And therefore it is called, The Double Golden Rule, or Double Rule of Three.

The Double Rule Direct is, when the sixth Term or Number sought, is found by two Operations, both of them in Direct Proportion.

Example 1. If a Hundred Pounds gain six Pounds Interest in twelve Months; how much will three Hundred Pounds gain in nine Months at the same Rate.

First 100*l.* : 6*l.* :: 300*l.* : 18*l.*
6

100) 1800 (18*l.* The Interest of 300*l.* for twelve Months.

Months. *Months.*
Then, 12 : 18*l.* :: 9 : 13*l.* 10*s.*
9

12) 162 (13*l.* 10*s.* The Answer required.

I suppose the Learner will easily conceive the Reason of these two Operations. For, first it is plain by Direct Proportion, that if 100*l.* gain 6*l.* in twelve Months, 300*l.* will gain 18*l.* in the same Time, and at the same Rate.

And

And by the ſame Rule it is plain, that if 12 Months will produce or give 18*l.* Intereſt for 300*l.* then 9 Months muſt needs give $13\frac{1}{2}$ for the ſame Sum, *viz.* 300*l.*

The Double Rule of Three Inverſe is, when the ſixth Term or Number ſought is found at two Operations (as before). But one of them requires an Anſwer in Reciprocal Proportion.

Queſtion 2. If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days, How many Days will 21 Buſhels ſerve 16 Horſes at the ſame Rate of feeding?

This *Queſtion* being parted into two Poſitions, the firſt will be thus:

If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days, How many Days will 21 Buſhels ſerve them?

Here it is plain that 21 Buſhels will ſerve them longer than 6 Buſhels; therefore the firſt Poſition falls in Direct Proportion.

	Buſh.		*Days.*		*Buſh.*		*Days.*
Thus,	6	:	8	::	21	:	28
					8		
					6) 168 (28 Days		

That is, if 6 Buſhels will ſerve 4 *Horſes* 8 *Days,* 21 *Buſhels will ſerve them* 28 *Days.*

The next Poſition muſt be to find how long the 21 Buſhels will ſerve 16 Horſes at the ſame Rate of feeding: it is plain, that 21 Buſhels cannot ſerve 16 Horſes ſo many Days as they will ſerve 4 Horſes; therefore this ſecond Poſition falls in Reciprocal Proportion.

	Horſes.		*Days.*		*Horſes.*		*Days.*	
Thus,	4	:	28	::	16	:	7	The Anſwer required.

After the like Manner any Queſtion in the Double Rule of Three may be anſwered by two ſingle Poſitions, if Care be taken in ſtating them right, *viz.* Whether their Operation muſt be performed by the ſingle Rule Direct, or Inverſe.

But all Queſtions in this Double Rule, where five Numbers are propoſed to find a ſixth, may more eaſily and readily be anſwered by one general Theorem; which compriſeth both the Direct and Inverſe Rules; without conſidering either of them being deduced from the ſingle Operations before-going.

But firſt you muſt carefully note, that in all Queſtions of this Nature, three of the five propoſed Terms are always conditional

and

and ſuppoſed; and that the other two move the Queſtion. As for Inſtance in *Example* 1.

Viz. If 100*l.* will gain 6*l.* in 12 Months; theſe three Terms are only ſuppoſed or conditional. Then comes the Queſtion; What will 300*l.* gain in nine Months? Now in Order to raiſe the general Theorem, let us ſuppoſe, inſtead of Numbers theſe Letters.

$$\textit{Viz.}\ \text{Let}\ \left\{\begin{array}{ll} P=100. & \text{The } \textit{Principal.} \\ T=\ 12. & \text{The } \textit{Time.} \\ G=\ \ 6. & \text{The } \textit{Gain.} \end{array}\right\} \left\{\begin{array}{l} \text{In the Suppoſition} \\ \text{of any propoſed} \\ \text{Queſtion.} \end{array}\right.$$

$$\text{And,}\ \left\{\begin{array}{ll} p=300. & \text{The } \textit{Principal.} \\ t=\ \ 9. & \text{The } \textit{Time.} \\ g=13{,}5 & \text{The } \textit{Gain.} \end{array}\right\} \left\{\begin{array}{l} \text{The Three Terms} \\ \text{wherein the Que-} \\ \text{ſtion lies.} \end{array}\right.$$

$$\text{Then } P : G :: p : \frac{Gp}{P} = \left\{\begin{array}{l} \text{The Product of the two Means divided by} \\ \text{the firſt Extream.} \end{array}\right.$$

$$\text{That is, } 100 : 6 :: 300 : \frac{300 \times 6}{100} = 18. \left\}\left\{\begin{array}{l} \text{Which is the} \\ \text{firſt Part of the} \\ \text{Queſtion.} \end{array}\right.\right.$$

$$\begin{array}{l} \text{Then } T : \frac{Gp}{P} :: t : g \\ \textit{Viz.}\ \ 12 : 18 :: 9 : 13{,}5 \end{array} \left\}\left\{\begin{array}{l} \text{Which is the} \\ \text{ſecond Part of} \\ \text{the Queſtion.} \end{array}\right.\right.$$

$$\textit{Ergo } Tg = \frac{GPt}{P} \left\}\left\{\begin{array}{l} \text{That is, the Product of the Extreams is equal to} \\ \text{that of the Means.} \end{array}\right.\right.$$

Conſequently, $TgP = Gpt$ is the *Theorem.*

This *Theorem* affords two Rules, by which all Queſtions in this Double Rule of Three, or rather of five Numbers, may be reſolved; due Regard being had to the true placing down of the propoſed Terms, which muſt be thus:

Always place the three conditional Terms in this Order; let that Number which is the principal Cauſe of Gain, Loſs, or Action, &c. *(viz. P.)* be put in the firſt Place; that Number which denotes the Space of Time, or Diſtance of Place, &c. *(viz. T.)* be put in the ſecond Place. And that Number which is the Gain, Loſs, or Action, &c. *(viz. G.)* be put in the third Place. Now according to theſe Directions, the conditional Terms of the laſt Queſtion will ſtand thus; *P. T. G.*

That done, place the other two Terms which move the Queſtion, underneath thoſe of the ſame Name,

$$\text{Thus, } \left\{\begin{array}{l} P.\ T.\ G. \\ p.\ t. \end{array}\right.$$

Then

Then if the Blank or Term ſought, fall under the third Place, as in this Queſtion,

It will be $\left\{\frac{G\,p\,t}{T\,P}=g.\right.$ Which gives this Rule.

Rule 1. { *Multiply the three laſt Terms together for a Dividend, and the two firſt together for a Diviſor; the Quotient ariſing from them will be the ſixth Term.*

That is, in our propoſed *Example* 1.

Thus $6\times300\times9=16200$ the Dividend.
And $100\times12=1200$ the Diviſor.
Then 1200) 16200 ($13\frac{1}{2}$ the Anſwer, as before.

But if the Blank or Term ſought, fall under the firſt Place, then

It will be $\left\{\frac{T\,g\,P}{t\,G}=p.\right.$

Or if the Blank fall under the ſecond Place,

It will be $\left\{\frac{T\,g\,P}{G\,p}=t.\right.$ Either of theſe gives this Rule.

Rule 2. { *Multiply the firſt, ſecond, and laſt Terms together for a Dividend, and the other two together for a Diviſor; the Quotient ariſing from them will be the ſixth Term.*

And becauſe our *Example* 2. falls under the Conſideration both of Direct and Reciprocal Proportion, let it be here propoſed again.

Viz. If 6 Buſhels of Oats will ſerve 4 Horſes 8 Days; How many Days will 21 Buſhels ſerve 16 Horſes, &c.

If the Terms of this Queſtion be placed down as before directed, they will ſtand

	Horſes.		*Days.*		*Buſhels.*	
Thus {	4	.	8	.	6	Terms in the Suppoſition.
	16				21	

Here the Blank falls under the ſecond Place, therefore it muſt be found by the ſecond Rule.

Thus $4\times8\times21=672$ the Dividend.
And $16\times6=96$ the Diviſor.
Then 96) 672 (7 the Anſwer, as before.

Quest. 3. What Principal or Stock will gain 20*l.* in 8 Months at 6 *per Cent. per Annum?*

Prin.	*Time.*	*Gain.*	
100 .	12 .	6	Terms in the Supposition.
.	8 .	20	

In this Question the Blank falls under the first Place, therefore it must be found by the second Rule.

Thus $100 \times 12 \times 20 = 24000$ the Dividend.
And $8 \times 6 = 48$ the Divisor.
Then 48) 24000 (500*l.* the Answer required.

The Proof of all Questions in this Double Rule of five Numbers, is best performed by varying the Question; *viz.* by stating it in another Order, as in the last *Example:* Thus,

If 100*l.* gain 6*l.* in 12 Months, what will 500*l.* gain in 8 Months?

The Answer to this Question must be 20*l.* if the Work of the last *Example* be true.

	Prin.	*Time.*	*Gain.*	
Stated thus	100 .	12 .	6	then, *per* Rule 1.
	500 .	8		

$500 \times 8 \times 6 = 24000$. And $100 \times 12 = 1200$.
Then 1200) 24000 (20*l.* the Answer, &c.

Quest. 4. If 2 Men can do 12 Rods of Ditching in 6 Days, How many Rods may be done by 8 Men in 24 Days, at the same Rate of working?

Answer, 192 Rods.

Quest. 5. If the Carriage of 5 *C.* 3 *qrs. Weight*, 150 Miles, cost 3*l.* 7*s.* 4*d.* What must be paid for the Carriage of 7 *C.* 2 *qrs.* 25 lb. *Weight*, 64 Miles, at the same Rate?

Answer, 1*l.* 18*s.* 7½*d.*

Quest. 6. If 8 Men deserve 2*l.* Wages for 5 Days Work, How much will 32 Men deserve for 24 Days, at the same Rate?

Answer, 38*l.* 8*s.*

Quest. 7. Suppose a hundred Pounds would defray the Expences of five Men for twenty-two Weeks and six Days, How long would twelve Men be in spending of one hundred and fifty Pounds, at the same Rate?

Answer, 14 Weeks and 2 Days.

CHAP.

CHAP. VIII.

Of Trading in Company, usually called the RULE OF FELLOWSHIP; *also* BARTERING, *and* EXCHANGING *of Coins*, &c.

THE *Rule of Fellowship* is that by which the Accompts of several Partners trading in a Company, are so adjusted or made up, that every Partner may have his just Part of the Gain, or sustain his just Part of the Loss; according to the Proportion or Share of Money he hath in the Joint-Stock: Now this falls under two Considerations, called the *Single* and *Double Rules of Fellowship*.

Sect. 1. *The* SINGLE RULE OF FELLOWSHIP; viz. *That without* Time.

BY the *Single Rule of Fellowship* is adjusted the Accompts of those Partners that put all their several and perhaps different Sums of Money, into a common Stock at one and the same Time; and therefore it is usually called the *Rule of Fellowship* without Time: Now all Questions of this Nature are answered by so many several Operations in the *Rule of Three Direct*, as there are Partners in the Stock.

For, as the total Sum of Money in the Stock is in Proportion to the whole Gain, or Loss; so is every Man's particular Part of that Stock, to his particular Share of that Gain, or Loss.

Quest. 1. Three Partners, suppose *A*, *B*, and *C*, make a Joint-Stock of 96*l.* in this Manner.

A, puts in 24*l.* *B*, puts in 32*l.* and *C*, puts in 40*l.* with this 96*l.* they Trade, and gain 12*l.* It is required to find each Man's true Part of that Gain.

The Operation will stand, thus

$$96l. : 12l. :: \left\{\begin{array}{l} 24l. : 3l. = A\text{'s} \\ 32l. : 4l. = B\text{'s} \\ 40l. : 5l. = C\text{'s} \end{array}\right\} \text{Part of the Gain.}$$

Proof $3l. + 4l. + 5l. = 12l.$ the whole Gain.

That is, if the Sum of each Man's particular Gain, amount to the whole Gain, the Work is true; if not, some Error is committed which must be found out.

Note, These Operations will be very much abbreviated, if you work them by *Theorem* 2. page 87. For here 96 is a common Antecedent, and 12 is the common Consequent in all the three Proportions.

Therefore 96 : 12 :: 1 : 0,125 is a common Multiplicator.

$$\text{Then}\ \left.\begin{array}{r}24\\32\\40\end{array}\right\}\times 0{,}125=\left\{\begin{array}{l}3l.\\4l.\\5l.\end{array}\right\}\ \text{for}\ \left\{\begin{array}{l}A,\\B,\\C,\end{array}\right\}\ \text{as before.}$$

Now this Method is more readily performed than the other, especially when the Partners are many; because one single Division serves for all the Work.

Quest. 2. Three Merchants, *A*, *B*, and *C*, freight a Ship with 248 Tun of Wine: Thus, *A*, loaded 98 Tun, *B*, 86 Tun, and *C*, 64 Tun. By Extremity of Weather the Seamen were forced to cast or throw 93 Tun of it over-board. How much of this Loss must each Merchant sustain?

First 248 : 93 :: 1 : 0,375 the common Multiplier.

$$\text{Then}\ \left.\begin{array}{r}98\\86\\64\end{array}\right\}\times 0{,}375=\left\{\begin{array}{l}36{,}75\ \text{for}\ A\text{'s}\\32{,}25\ \text{for}\ B\text{'s}\\24{,}00\ \text{for}\ C\text{'s}\end{array}\right\}\ \text{Loss.}$$

Proof 93,00=the whole Loss.

Now if the Question were to find how much of the remaining Wine that was saved, belongs to *A*, to *B*, and to *C*.

$$\text{Then}\ \left.\begin{array}{r}98-36{,}75=61{,}25\\86-32{,}25=53{,}75\\64-24{,}00=40{,}00\end{array}\right\}\ \text{belongs to}\ \left\{\begin{array}{l}A.\\B.\\C.\end{array}\right.$$

That is, *A*, ought to have 61 Tun and 63 Gallons. *B*, ought to have 53 Tun and 189 Gallons. And *C*, ought to have 40 Tun of what was left.

Quest. 3. Suppose 6 Men, *viz.* *A*, *B*, *C*, *D*, *E*, and *F*, make a Joint-Stock of 2558*l*.

$$\begin{array}{rl}
 & \begin{array}{rcl}l. & s. & \textit{Decimals.}\end{array}\\
\text{Thus}\ \left.\begin{array}{r}A\\B\\C\\D\\E\\F\end{array}\right\}\text{puts in} & \left\{\begin{array}{rcl}654 & . & 10=654{,}50\\543 & . & 15=543{,}75\\480 & . & 00=480{,}00\\254 & . & 10=254{,}50\\365 & . & 05=365{,}25\\260 & . & 00=260{,}00\end{array}\right.
\end{array}$$

The whole Stock 2538 . 00=2558,00 according to the *Quest.*

With

With this Stock of 2558*l.* they Trade eighteen Months, and gain 831*l.* 7*s.* It is required to find every Man's Part or Share of that Gain.

Note, *Although the Time of Trading, viz. eighteen Months, be mentioned in the Question, yet it is no Way concerned in answering of it; as you may observe in the following Work.*

First, 2558*l.* : 831,35*l.* : : 1*l.* : 0,325 Decimal Parts.
Consequently, 1*l.* : 0,325 : : 654,5 : 212,7125. That is,

$$\left.\begin{array}{r}654{,}50\\543{,}75\\480{,}00\\254{,}50\\365{,}25\\260{,}00\end{array}\right\}\times 0{,}325=\left\{\begin{array}{r}212{,}71250\\176{,}71875\\156{,}00000\\82{,}71250\\118{,}70625\\84{,}50000\end{array}\right\}\text{for}\left\{\begin{array}{l}A.\\B.\\C.\\D.\\E.\\F.\end{array}\right.$$

$$\begin{array}{lcl}
 & & \begin{array}{rrrrr}\textit{l.} & \textit{parts.} & \textit{l.} & \textit{s.} & \textit{d.}\end{array}\\
\text{That is, }\left.\begin{array}{r}A\\B\\C\\D\\E\\F\end{array}\right\} & \text{gains} & \left\{\begin{array}{rrrr}212{,}71250= & 212 & .\ 14 & .\ 03\\176{,}71875= & 176 & .\ 14 & .\ 04\tfrac{1}{4}\\156{,}00000= & 156 & .\ 00 & .\ 00\\82{,}71250= & 82 & .\ 14 & .\ 03\\118{,}70625= & 118 & .\ 14 & .\ 01\tfrac{1}{2}\\84{,}50000= & 84 & .\ 10 & .\ 00\end{array}\right.\\
\hline
\text{Proof.} & \text{Sum} & 831{,}35\quad =831\ .\ 07\ .\ 00
\end{array}$$

I have omitted resolving this Question according to the usual Method (as before directed) of finding every Man's particular Part of the Gain by the *Golden Rule*, as in the first Work of *Example* 1. leaving that for the Learner's Practice.

Sect. 2. *The* Double Rule of Fellowship; *or that with* Time.

THIS is usually called the *Double Rule of Fellowship*, because every particular Man's Money is to be considered with Relation to the Time of it's Continuance in the Joint-Stock.

Question 1. *A*, and *B*, join in Partnership upon these Terms, *viz.* *A*, agrees to lay down 100*l.* and to employ it in Trade 3 Months: Then *B*, is to lay down his 100*l.* and with the whole Stock of 200*l.* they are to trade 3 Months more. Now at the End of that Time, they find their whole Gain to be 21*l.* It is required to know what each Man's Part of the Gain ought to be, according to his Stock, and the Time of employing it.

Here

Here it is but reasonable to conclude, that *A*, ought to gain more than *B*, notwithstanding their Stocks of Money are equal; because *A* employed his Money a longer Time than *B*.

Now for resolving of this Question, let us suppose *A*'s 100*l*. employed the first 3 Months to gain Z=a Sum as yet unknown; then it must gain 2 Z in 6 Months; and to find what *B*, must gain, it will be,

l.	*Months.*		
100	. 6 .	2 Z=*A*'s Gain	*per Rule* 1. *Page* 97.
100	. 3 .	to *B*'s Gain	

Ergo $\frac{100\times3\times2Z}{100\times6}=$ *B*'s Gain.

But *A*'s Gain added to *B*'s Gain must=21*l*. the whole Gain by the Question.

Therefore $2\,Z+\frac{100\times3\times2\,Z}{100\times6}=21l.$

That is, $100\times6\times2\,Z+100\times3\times2\,Z=21\times100\times6$.
Which contracted is, $900\times2\,Z=21\times600$.

Consequently, $2\,Z=\frac{21\times600}{900}$ which gives the following Analogy.

Viz. 900 : 21 : : 600 : 2 Z=14*l*. for *A*'s Gain.
And 900 : 21 : : 100×3=300 : 7*l*. for *B*'s Gain.

Now this Way of arguing hath not only resolved the present Question, but it also affords (and demonstrates) a general Rule for resolving all Questions of this Nature, be the Partners never so many.

Rule. { *Multiply every particular Man's Stock, with the Time it is employed, then it will be, as the Sum of all those Products, is to the whole Gain (or Loss). So is every one of those Products; to its proportional Part of that whole Gain (or Loss).*

Question 2. Three Merchants *A*, *B*, and *C*, enter into Partnership, thus; *A* puts into the Stock 65*l*. for 8 Months; *B* puts in 78*l*. for 12 Months; and *C* puts in 84*l*. for 6 Months. With these they traffick, and gain 166*l*. 12*s*. It is required to find each Man's Share of the Gain, proportionable to the Stock and Time of employing it.

$$\left.\begin{array}{l}1.\ A\text{'s}\\2.\ B\text{'s}\\3.\ C\text{'s}\end{array}\right\}\text{Stock}\left\{\begin{array}{l}65l.\times 8\\78l.\times 12\\84l.\times 6\end{array}\right\}\textit{Months, the Time it was employed}=\left\{\begin{array}{r}520\\936\\504\end{array}\right.$$

The Sum of those Products is, 1960

Then, according to the Rule, the several Proportions will stand thus,

$$1960 : 166,6 :: \left\{\begin{array}{l}520 : 44,20 = 44l.\ 4s.\ 0d.\\936 , 79,56 = 79l.\ 11s.\ 2\frac{1}{2}d.\\504 ; 42,84 = 42l.\ 16s.\ 9\frac{1}{2}d.\end{array}\right\}\text{for}\left\{\begin{array}{l}A.\\B.\\C.\end{array}\right.$$

The whole Gain $= 166l.\ 12s.\ 0d.$

Or you may work as in some of the former *Examples*, viz. by finding the proportional Part of the Gain due to one Pound, &c.

Thus $1960 : 166,6 :: 1 : 0,085$ the common Multiplier.

$$\text{Then}\ \left.\begin{array}{r}520\\936\\504\end{array}\right\}\times 0,085 = \left\{\begin{array}{l}44,2\\79,56\\42,84\end{array}\right\}\text{for}\left\{\begin{array}{l}A.\\B.\\C.\end{array}\right\}\text{\&c. As before.}$$

Question 3. Six Merchants, *viz.* *A*, *B*, *C*, *D*, *E*, and *F*, enter into Partnership, and compose a Joint-Stock in this Manner:

$$\textit{Viz.}\left\{\begin{array}{l}A\\B\\C\\D\\E\\F\end{array}\right\}\text{puts in}\left\{\begin{array}{rcl}l. & & s.\\64 & . & 10\\78 & . & 15\\100 & . & 00\\80 & . & 10\\74 & . & 12\\125 & . & 15\end{array}\right\}\text{for}\left\{\begin{array}{l}4\frac{1}{2}\\6\\8\frac{1}{4}\\12\\9\frac{1}{2}\\7\end{array}\right\}\text{Months.}$$

They traffick, and gain $258l.\ 18s.\ 4\frac{1}{2}d.$ It is required to find every Man's Share of the Gain, according to the Stock and Time it was employed.

The several Stocks of Money, and their respective Times being first brought into Decimals, and then multiplied together, will produce these following Products.

$$\left.\begin{array}{l}A\text{'s}\\B\text{'s}\\C\text{'s}\\D\text{'s}\\E\text{'s}\\F\text{'s}\end{array}\right\}\text{Stock}\left\{\begin{array}{l}l.\quad \textit{Months.}\\64,50\times 4,50\\78,75\times 6,00\\100,00\times 8,25\\80,50\times 12,00\\74,6\ \times 9,50\\125,75\times 7,00\end{array}\right\}\textit{the Time it was employed}=\left\{\begin{array}{r}290,25\\472,50\\825,00\\966,00\\708,70\\880,25\end{array}\right.$$

The Sum of those Products $= 4142,70$

Then

Then if you work by the common Way; it will be 4142,7 : 258,91875 : : 290,25 : 18,140625=18*l.* 2*s.* 9¾*d.* for *A*'s Part of the Gain; and so on for the rest.

But if you work by the easiest Way, *viz.* by finding the proportional Part of the Gain due to one Pound.

Thus 4142,7 : 258,91875 : : 1 : 0,0625.

$$\text{Then}\quad \left.\begin{array}{r} 290,25 \\ 472,50 \\ 825,00 \\ 966,00 \\ 708,70 \\ 880,25 \end{array}\right\} \times 0,0625 = \left\{\begin{array}{l} 18,140625 = 18 \;.\; 02 \;.\; 09\tfrac{1}{4} \\ 29,531250 = 29 \;.\; 10 \;.\; 07\tfrac{1}{2} \\ 51,562500 = 51 \;.\; 11 \;.\; 03 \\ 60,375000 = 60 \;.\; 07 \;.\; 06 \\ 44,293750 = 44 \;.\; 05 \;.\; 10\tfrac{1}{2} \\ 55,015625 = 55 \;.\; 00 \;.\; 03\tfrac{3}{4} \end{array}\right\} \text{for} \left\{\begin{array}{l} A \\ B \\ C \\ D \\ E \\ F \end{array}\right.$$

(columns: *l.* *s.* *d.*)

The whole Gain=258 . 18 . 04½

These few *Examples* being well understood, are sufficient to shew the whole Business of Fellowship, &c.

Sect. 3. *Of* Bartering.

WHEN Merchants, or Tradesmen, exchange one Commodity for another, it is called *Bartering*; and the only Difficulty in this Way of dealing, lies in duly proportioning the Commodities to be exchanged, so as that neither Party may sustain Loss.

Question 1. Two Merchants, *A*, and *B*, *Barter*; *A* would exchange 5 *C.* 3 *qrs.* 14 lb. of *Pepper*, which is worth 3*l.* 10*s.* *per C.* with *B* for *Cotton*, worth 10*d.* *per pound* Weight; how much *Cotton* must *B* give to *A* for his *Pepper?*

Note, *In order to the resolving of this Question (and all other Questions of this Nature) you must first find, by the Rule of Three (or otherwise) the true Value of that Commodity whose Quantity is given (which in this Question is Pepper). And then find how much of the other Commodity will amount to that Sum, at the Rate proposed.*

First 5 *C.* 3 *qrs.* 14 lb. =5,875 }
And 3*l.* 10*s.* 0*d.* =3,500 } in Decimals.

Then 1 : 3,5 : : 5,875 : 20,5625=20*l.* 11*s.* 3*d.* the true Value of the *Pepper*.

Next, It is easy to conceive, that *A* ought to have as much *Cotton* at 10*d.* *per Pound*, as will amount to 20*l.* 11*s.* 3*d.* which may be thus found;

10*d.* : 1 lb. : : 20*l.* 11*s.* 3*d.*=4235*d.* : 493,5 lb.

That

That is 4 *C.* 1 *qr.* 17⅞ *pound* of *Cotton*. And ſo much *B* muſt give to *A* in exchange for his 5 *C.* 3 *qrs.* 14 lb. of *Pepper*.

Queſtion 2. Two Merchants, *A* and *B*, barter thus; *A* hath 86 Yards of Broad-cloth worth 9*s.* 2*d.* *per Yard* ready Money: but in Barter he will have 11*s.* *per Yard*. *B* hath Shalloon worth 2*s.* 1*d.* *per Yard* ready Money; it is required to find how many Yards of the Shalloon *B* muſt give to *A* for his Cloth, to make his Gain in the Barter equal to that of *A*'s.

The Method of reſolving this, and the like Queſtions, differs a little from the laſt Caſe; for in this you muſt firſt find what Advance *B* ought to make *per Yard* upon his Shalloon, in proportion to what *A* hath done upon a Yard of his Cloth.

$$\text{Thus}\left\{\begin{array}{llllllllll} s. & d. & d. & s. & d. & s. & d. & d. & s. & d.\ \ d. \\ 9 & 2=110 & : & 11=132 & :: & 2 & 1=25 & : & 2 & 6=30 \end{array}\right.$$

the advanced Price for a Yard of *B*'s Shalloon. Then proceed as before in the laſt *Example*.

Thus 1 Yard : 11*s.* :: 86 Yards : 946*s.*=47*l.* 6*s.* the advanced Value of all the Cloth.

Next, If 2*s.* 6*d.* will buy one Yard of Shalloon, at its advanced Price, how many Yards will 47*l.* 6*s.* buy.

Thus 2,5 : 1 :: 946 : 378,4 Yards.

That is, *B* muſt give 378⅖ Yards of his Shalloon to *A*, for his 86 Yards of Broad-cloth.

Theſe two Examples are ſufficient to ſhew the Learner, that the Method of bartering, or exchanging Commodities for Commodities, wholly depends upon a clear Underſtanding of the *Golden Rule*; which indeed is ſo called, becauſe of its univerſal Uſe.

Sect. 4. *Of* Exchanging Coins.

EXchanging the Coins of one Country for thoſe of another, is like the Buſineſs of bartering Commodities. That is, it conſiſts in finding what Sum of one Country Coin will be equal in Value to any propoſed Sum of another Country Coin. And in order to perform that, it will be very neceſſary to have a true Account at all Times of the juſt Value of thoſe foreign Coins which are to be exchanged, as they are compared in Value with our *Engliſh* Coin.

I ſay, at all Times, becauſe the Par of Exchange (as the Merchants call it) differs almoſt every Day from *London* to other Countries. That is, it riſes and falls, according as Money is plenty or ſcarce; or according to the Time allowed for Payment of the Money in Exchange, &c.

Those that desire to be fully satisfied in the common Values of foreign Coins, Weights, Measures, &c. may find them in a Book called the *Merchants Map of Commerce*, which for Brevity sake, I have omitted transcribing, and only collected these few of Coin.

	Foreign Coins.	*English Coin.* *l.*	*s.*	*d.*
French Coin.	A *Denier*=	0	0	$\frac{3}{40}$
	12 *Deniers*=1 *Soulz*=	0	0	$\frac{9}{10}$
	12 *Soulz*=1 *Livre*=	0	1	6
	3 *Livres*=1 *Crown*=	0	4	6
Low Country Coin.	A *Stiver*=	0	0	$1\frac{1}{5}$
	6 *Stivers*=1 *Flemish Shilling*=	0	0	$7\frac{1}{5}$
	20 *Stivers*=1 *Gilder*=	0	2	
	10 *Gilders*=$33\frac{1}{3}$ *Shillings* or a *Flemish Pound* =	1	0	
	A *Embden Dollar*=	0	2	$3\frac{3}{5}$
	A *Campen Dollar*=	0	2	$7\frac{1}{5}$
	A *Zealand Dollar*=	0	3	
	A *Lyons Dollar*=	0	4	
	A *Specie Dollar*=	0	5	
	A *Duccatoon*=	0	6	$3\frac{3}{5}$
Germany.	A *Rixdollar* of the *Empire*=	0	4	$5\frac{3}{4}$
	A *Gilder* of *Nuremberg*=	0	7	1
In Italy *and* Spain.	The *Liver* at *Leghorn*=	0	0	9
	Florence Crown Current=	0	5	3
	Venice Ducat de Banco=	0	4	4
	The *Current Ducat*=	0	3	4
	The *Naples Ducat*=	0	5	
	The *Cadiz Ducat*=	0	5	$6\frac{1}{4}$
	The *Barcelona Ducat*=	0	6	
	The *Valencia Ducat*=	0	5	3
	The *Bergonia Ducat*=	0	4	4
	The *Portugal Testoon*=	0	1	3
	The Piece of Eight=	0	4	6

Note, The *English* generally reckon their Exchange with other Countries by Pence, *viz.* other Countries value their Crowns, Dollars, or Ducats, &c. by *English* Pence. Except with some Parts of the *Low-Countries*, with whom the Exchange is in Pounds Sterling.

Quest. 1. How many Dollars at 4*s.* 6*d. per Dollar*, may one have for 162*l.* 18*s.* Answer, 724 Dollars.

Thus

Thus 162*l*. 18*s*.=3258*s*. and 4*s*. 6*d*.=4,5*s*.
Then 4,5 : 1 :: 3258 : 724 the Answer.

Quest. 2. How many *Saragossa* Ducats, of 5*s* 6*d*. the Ducat, may be had for 275 *Bergonia* Ducats at 4*s*. 4*d*. the Piece?
Answer, 216 and 3*s*. 8*d*. over.

Thus 5*s*. 6*d*.=66*d*. and 4*s*. 4*d*.=52*d*.
Then 275×52=14300*d*.=275 Ducats.
Consequently 66) 14300 (216$\frac{2}{3}$ the Answer required.

Quest. 3. A Traveller would change 233*l*. 16*s*. 8*d*. Sterling Money; for *Venice* Ducats at 4*s*. 9$\frac{1}{2}$*d*. *per Ducat*; How many Ducats must he have? Answer, 976 Ducats.

Thus 4*s*. 9$\frac{1}{2}$*d*.=57,5*d*. and 233*l*. 16*s*. 8*d*.=56120*d*.
Then 57,5*d*.) 56120*d*. (976 the Answer required.

Quest. 4. A Cashier hath received 759 Ducats at 7*s*. 6*d*. *per Ducat*; and 579 Dollars at 4*s*. 8*d*. *per Dollar*: Which he would exchange for *Flemish* Marks at 14*s*. 3*d*. *per* Piece: How many ought he to have?
Answer, 589 Marks, and 15*d*. over.

For 7*s*. 6*d*.=90*d*. and 4*s*. 8*d*.=56*d*.

Then $\begin{cases} 759\times90=68310d. \text{ the Value of the Ducats.} \\ 579\times56=32424d. \text{ the Value of the Dollars.} \end{cases}$

their Sum=100734*d*.
And 14*s*. 3*d*.=171*d*. the *Flemish* Mark in Pence.
Consequently 171) 100734 (589 &c. the Answer required.

Quest. 5. A Bill of Exchange was accepted at *London* for the Payment of 400*l*. Sterling, for the like Value delivered in *Amsterdam*, at 1*l*. 13*s*. 6*d*. for 1*l*. Sterling; How much Money was delivered at *Amsterdam*?
Answer, 670*l*. *Flemish*.

For 1*l*.=240*d*. and 1*l*. 13*s*. 6*d*.=402*d*.
Then 242 : 402 :: 400 : 670 the Answer required.

Quest. 6. When the Exchange from *Antwerp* to *London* is at 1*l*. 4*s*. 7*d*. *Flemish*, for 1*l*. Sterling; How many Pounds Sterling must be paid at *London*, to balance 236*l*. *Flemish* at *Antwerp*.
Answer, 192*l*. Sterling.

Thus 1*l*. 4*s*. 7*d*.=295*d*. and 1*l*.=240*d*.
Then 295 : 240 :: 236 : 192 the Answer.

Queſt. 7. A Merchant delivered at *London* 120*l.* Sterling to receive 147*l.* *Flemiſh* in *Amſterdam*; How much was 1*l.* Sterling valued at, in *Flemiſh* Money?

Anſwer, 1*l.* 4*s.* 6*d.*

Thus 120 : 147 :: 240*d.* : 294*d.*=1*l.* 4*s.* 6*d.* &c.

Queſt. 8. A Factor hath ſold Goods at *Cadiz* for 1468 Pieces of Eight, valued at 4*s.* 6½*d.* Sterling per Piece; How much Sterling Money do thoſe Pieces of Eight amount to?

Anſwer, 333*l.* 7*s.* 2*d.*

Thus: if 1=54,5*d.* then 1468×54,5=80006*d.* &c.

Queſt. 9. A Traveller would have an equal Number of Crowns at 5*s.* 6*d.* *per* Crown; and Dollars at 4*s.* 5*d.* *per* Piece; How many of each Sort may he have for 309*l.* 8*s.*?

Anſwer, 624 of each.

Thus 309*l.* 8*s.*=74256*d.*
And 5*s.* 6*d.*+4*s.* 5*d.*=119*d.*
Then 119) 74256 (624 the Anſwer required.

Queſt. 10. Suppoſe I would exchange 527*l.* 17*s.* 6*d.* for Dollars at 4*s.* 6*d.* a Piece, Ducats at 5*s.* 8*d.* a Piece, and Crowns at 6*s.* 1*d.* a Piece; and would have 2 Dollars for 1 Ducat, and 3 Dollars for 2 Crowns. How many of each Sort muſt I have?

Anſwer, 927 Dollars, 463½ Ducats, and 618 Crowns.

For $\left\{\begin{array}{l}54d.=1 \text{ Dollar.}\\ 68d.=1 \text{ Ducat.}\\ 73d.=1 \text{ Crown.}\end{array}\right\}$ *per Queſtion.*

And 126690*d.*=527*l.* 17*s.* 6*d.*

Now if the Crowns, Dollars, and Ducats, were to be equal in Number; then 73+54+68 muſt have been the Diviſor, by which 126690 muſt have been divided, and the Quotient would have been the Anſwer to the Queſtion. As in the laſt *Example*.

But here inſtead of their Sum, ſuch Parts of them muſt be taken as are aſſigned or limited by the Queſtion; that ſo the Number of ſome one of them may be found.

And becauſe there muſt be $\left\{\begin{array}{l}\text{2 Dollars for 1 Ducat, and}\\ \text{3 Dollars for 2 Crowns,}\end{array}\right.$

Therefore it will be ½ of a Ducat for one Dollar, and ⅔ of a Crown for one Dollar.

Conſequently,

Conſequently, $54 + \frac{68}{2} : + \frac{2}{3}$ of $73 = 136\frac{2}{3}$, or $\frac{410}{3}$ will be the Diviſor to find the Number of Dollars.
Thus $\frac{410}{3}$) 126690 (927 the Number of Dollars.
Then $\frac{1}{2}$ of $927 = 463\frac{1}{2}$ is the Number of Ducats.
And $\frac{2}{3}$ of $927 = 618$ is the Number of Crowns.

Or if you pleaſe, you may form Diviſors to find either the Ducats or Crowns firſt: For if it be 2Dollars for 1 Ducat, and 3 Dollars for 2 Crowns, as before;
Then will 6 Dollars be for 3 Ducats, and ſix Dollars for 4 Crowns.

$$\text{Therefore, } \left\{ \begin{array}{l} \frac{3}{2} \text{ of a Dollar} \\ \frac{3}{4} \text{ of a Ducat} \end{array} \right\} \text{ will be for 1 Crown.}$$

Conſequently, $\frac{3}{2}$ of $54 : + \frac{3}{4}$ of $68 : + 73 = 205$ will be the Diviſor to find the Crowns firſt, &c.

Queſt. 11. A Caſhier is to receive 500*l.* He is offered Crowns at 6*s.* $1\frac{1}{2}$*d.* *per* Crown, which are worth but 6*s.* Or he may have Dollars at 4*s.* 5*d.* the Piece, which are worth but 4*s.* 4*d.* Which of theſe ſhall he receive to have the leaſt Loſs? And how much will he loſe in the Payment?

$$1 \left\{ \begin{array}{l} 1 \text{ Crown} = 72d. \\ 1 \text{ Dollar} = 52d. \end{array} \right\} \text{ according to the true Values.}$$

$$2 \left\{ \begin{array}{l} 1 \text{ Crown} = 73{,}5d. \\ 1 \text{ Dollar} = 53{,}0d. \end{array} \right\} \text{ the advanced Values.}$$

Now to find which will be the leaſt Loſs; find what the advanced Value of a Dollar ought to be in proportion to that of 1 Crown.
Thus $72 : 73{,}5 :: 52 : 53{,}083$ &c. But he may have Dollars at 53*d.* per Piece, therefore the Payment in Dollars will be the leaſt Loſs; *viz.* 53 is leſs than 53,083 &c.
Next, to find what the whole Loſs will be by receiving Dollars. Becauſe the 500*l.*=120000*d.* is advanced as much above the true Value, as 53*d.* is above 52*d.*: therefore ſay, If 53*d.* advance 1*d.*=53*d.*—52*d.*; what will 120000*d.* advance? *i. e.*

$$53d. : 1d. :: 120000d. : 2264\tfrac{8}{53}d. = 9l.\ 8s.\ 8\tfrac{8}{53}d. = \text{the Loſs.}$$

Queſt. 12. Suppoſe I exchange 4*l.* 10*s.* 10*d.* for 11 Crowns and 7 Dollars; and at another Time I have 4 Crowns and 3 Dollars for 1*l.* 15*s.* each being of the ſame Value with the firſt. What is the Value of a Crown, and of a Dollar?

First

First 11 Crowns+7 Dollars=1090*d.* } by the Question.
Second 4 Crowns+3 Dollars= 420*d.* }

Then in order to find the Value of 1 Crown, you must cast off the Dollars by making them of the same Number; Thus,

33 Crowns+21 Dollars=3270*d.* the first multipl. with 3.
28 Crowns+21 Dollars=2940*d.* the second multipl. with 7.

Then 5 Crowns=330*d.* being the Difference.
Consequently 5) 330 (66=5*s.* 6*d.* is the Value of 1 Crown.
And 4 Crowns=264*d.*
Then will 3 Dollars=420*d.*—264*d.*=156*d.*
Consequently 3) 156 (52*d.*=4*s.* 4*d.* the Value of 1 Dollar.

CHAP. IX.

Of Alligation.

WHEN it is required to mix several Sorts of Ingredients together; as different Sorts of Corn, Wines, Wool, Spices, or Metals; or to compose Medicines, &c. the Method of proportioning such Mixtures, is called the *Rule of Alligation*; and is divided into two Parts or Branches, called *Medial* and *Alternate.*

Sect. 1. *Of* Alligation Medial.

A*Lligation Medial,* is that by which the mean Rate or Price of any Mixture is found, when the particular Quantities of the Mixtures and Rates are given; and is thus performed.

First find the Sum of all the Quantities proposed to be mixed? And also the Sum of all their particular Rates.

Then the Proportion will be,

Rule { *As the Sum of all the Quantities: Is to the Sum of all their Rates :: So is any Part of the Mixture: To the mean Rate or Price of that Part.*

Quest. 1. Suppose 15 Bushels of Wheat at 5*s.* the Bushel, and 12 Bushels of Rye at 3*s.* 6*d.* the Bushel, were mixed together; What

What is the mean Rate or Price, it may be ſold for a Buſhel, without Loſs or Gain?

This Queſtion prepared as directed above will ſtand.

Thus { 15 Buſhels of Wheat at 5*s.* *per* Buſhel, comes to 900*d.*
12 Buſhels of Rye at 3*s.* 6*d.* each, comes to 504*d.*

27=their Sum. And their total Value=1404*d.*

Then 27 Buſhels : 1404*d.* : : 1 Buſhel : 52*d.*=4*s.* 4*d.* the Anſwer required.

Queſt. 2. A Grocer mixeth 36 Pounds of Tobacco, worth 1*s.* 6*d.* a Pound, with 12 Pounds of another Sort at 2*s.* a Pound, and 12 Pounds of a third Sort at 1*s.* 10*d.* the Pound. How may he ſell the Mixture *per* Pound?

	lb.		*s.*	*d.*		*d.*
First	36	at	1	6	*per* Pound amonnts to	648
	12	at	2	0		288
	12	at	1	10		264

60=the Number of Pounds. Their Value=1200

Then 60 lb. : 1200*d.* : : 1 lb. : 20*d.*=1*s.* 8*d.* the Anſwer required.

Queſt. 3. A Vintner mixeth 31 Gallons and a half of *Malaga* Sack worth 7*s.* 6*d.* the Gallon; with 18 Gallons of Canary at 6*s.* 9*d.* the Gallon; 13 Gallons and a half of Sherry at 5*s.* the Gallon; and 27 Gallons of White-Wine at 4*s.* 3*d.* the Gallon. It is required to find what one Gallon of this Mixture is worth.

	Gal.		*s.*	*d.*		*Pence.*
First,	$31\frac{1}{2}$	at	7	6	*per* Gallon, comes to	2835
	18	at	6	9		1458
	$13\frac{1}{2}$	at	5	0		810
	27	at	4	3		1377

90=the Numb. of *Gal.* Their Value=6480

Then 90 : 6480 : : 1 : 72*d*=6*s.* the Rate or Price of one Gallon, as was required.

¶ The Proof of all Operations in theſe Sort of Mixtures, is done by comparing the Value of all the Mixture (being ſold at the mean Rate) with the total Value of all the particular Quantities, ſuppoſing they had been ſold at their reſpective Rates unmixed; if thoſe Sums are equal, the Work is true.

Sect.

Sect. 2. *Of* Alligation Alternate.

ALligation Alternate, is that by which the particular Quantities of every Ingredient concerned in any Mixture are found; when the particular Rates of every one of thoſe Ingredients, and the mean Rate are given; being (as it were) the Converſe to Alligation Medial; as will appear by the following Operations, which admit of three Caſes.

Caſe I. The particular Rates of any Ingredients propoſed to be mixed, and the mean Rate of the whole Mixture being given. To find how much of each Ingredient is requiſite to compoſe the Mixture, when the whole Quantity, or any Part thereof, is not limited.

Queſt. 1. How much Wheat at 5*s.* the Buſhel, and Rye at 3*s.* 6*d.* the Buſhel, will compoſe a Mixture that may be ſold for 4*s.* 4*d.* the Buſhel.

Note, *In all Queſtions of this Nature, it will be convenient to place the mean Rate ſo, as that it may be eaſily compared with the particular Rates, in order to find every one of their Differences from the mean Rate, by Inſpection only.*

$$\text{Thus, the mean Rate}=52d. \left\{\begin{array}{ll}\text{Wheat} & 60d.\\ \text{Rye} & 42d.\end{array}\right.$$

Then take the ſeveral Differences between the mean Rate, and the particular Rates; ſetting down thoſe Differences alternately, and they will be the Quantities required.

$$\text{Thus } 52\left\{\begin{array}{l}60\\42\end{array}\right\}\left\{\begin{array}{l}10=52-42\\ 8=60-52\end{array}\right.$$

That is 52—42=10 for the Quantity of Wheat.

And 60—52= 8 for the Quantity of Rye, that will compoſe the Mixture required.

The Proof by *Alligation Medial.*

Add	10 Buſhels of Wheat at 60*d.* *per* Buſhel	=600*d.*
	8 Buſhels of Rye at 42*d.* *per* Buſhel	=336*d.*
	18=the Number of Buſhels.	=936*d.*

Then 18 : 936 : : 1 : 52*d.*=4*s.* 4*d.* the mean Rate.

Note, Altho' 10 and 8 do anſwer the Queſtion, as plainly appears by the Proof; yet they are not the only two Numbers; for this Queſtion, and all others of this Kind, will admit of various Anſwers, and all whole Numbers; for any two Numbers that are in the ſame Proportion to one another, as 10 is to 8, will as truly anſwer the Queſtion.

Viz.

$$Viz.\ 10 : 8 :: \begin{cases} 5 : 4 \\ 15 : 12 \\ 20 : 16 \\ 25 : 20 \end{cases} \text{\&c. ad infinitum.}$$

Quest. 2. A Grocer would mix three Sorts of Tobacco together, *viz.* One Sort of 18*d.* *per* ℔, another Sort of 22*d.* *per* ℔, and a third Sort of 2*s.* the ℔. How much of each Sort must he take, that the whole Mixture may be sold for 20*d.* the Pound?

Having set down the given Rates, as before, then find each of their Differences from the proposed Mean Rate, and place those Differences alternately. Thus,

$$\text{Mean Rate } 20 \begin{cases} 18 \\ 22 \\ 24 \end{cases} \begin{cases} 4 + 2 = 24 - 20 \text{ and } 22 - 20 \\ 2 = 20 - 18 \\ 2 = 20 - 18 \end{cases}$$

These Differences, *viz.* 6 . 2 . 2 are the Quantities required.

$$Proof \begin{cases} 6\text{℔ of Tobacco at } 18d. \\ 2\text{℔ at } 22d. \\ 2\text{℔ at } 24d. \end{cases} \text{the Pound come to} \begin{cases} 108 \\ 44 \\ 48 \end{cases} d.$$

10 = the Number of Pounds. Their Value = 200 *d.*

Then 10) 200 (20 the Mean Rate.

Or indeed any three Numbers that have the same Ratio to one another as 6 and 2 have, will answer the Question.

$$\text{That is, } 6 : 2 :: \begin{cases} 9 : 3 \\ 12 : 4 \\ 15 : 5 \end{cases} \text{\&c.}$$

But if only one of the three given Rates had been greater than the Mean Rate; as suppose 14*d.* *per* Pound, 18*d.* *per* Pound, and 24*d.* *per* Pound, and the Mean Rate 20*d.* as before. Then their Differences must have been placed,

$$\text{Thus, } 20 \begin{cases} 14 \\ 18 \\ 24 \end{cases} \begin{cases} 4 \\ 4 \\ 6 + 2 \end{cases} \text{\&c. as before.}$$

Quest. 3. A Vintner would make a Mixture of Malaga, worth 7*s.* 6*d.* *per* Gallon, with Canary at 6*s.* 9*d.* *per* Gallon, Sherry at 5*s.* *per* Gallon, and white Wine at 4*s.* 3*d.* *per* Gallon; What Quantity of each Sort must he take, that the Mixture may be sold for 6*s.* *per* Gallon?

In all Questions of this Kind, wherein it is required to mix four Things together, two of them having their Prices greater, and two lesser than the Mean Rate: you must always alligate or

compare a greater and lesser Price with the Mean Price, setting down their Differences alternately, as in the first *Example* of this *Section*.

$$\text{Thus, Mean Rate}=72d.\left\{\begin{array}{l}\left.\begin{array}{l}\text{Malaga } 90d.\\ \text{White } 51d.\end{array}\right\}\\ \left.\begin{array}{l}\text{Sherry } 60d.\\ \text{Canary } 81d.\end{array}\right\}\end{array}\right.\begin{array}{l}\left\{\begin{array}{l}21=72-51\\ 18=90-72\end{array}\right.\\ \left\{\begin{array}{l}9=81-72\\ 12=72-60\end{array}\right.\end{array}$$

Hence 21 Gallons of Malaga, 12 of Canary, 9 of Sherry, and 18 of White, will compose the Mixture required.

$$\text{Or thus, } 72\left\{\begin{array}{l}\text{Malaga } 90d.\\ \text{Sherry } 60d.\\ \text{Canary } 81d.\\ \text{White } 51d.\end{array}\right\}\left\{\begin{array}{l}12 \text{ Malaga}\\ 18 \text{ Sherry}\\ 21 \text{ Canary}\\ 9 \text{ White}\end{array}\right\}\text{will, \&c.}$$

Either of these Mixtures equally answer the Question, which may easily be tried as before in the last, &c.

Case II. The particular Rates of all the Ingredients proposed to be mixed, the Mean Rate of the whole Mixture, and any one of the Quantities to be mixed being given: Thence to find how much of every one of the Ingredients is requisite to compose the Mixture.

Note, This is usually called *Alligation Partial*.

Quest. 4. How much Wheat at 5*s*. the Bushel, must be mixed with 12 Bushels of Rye, at 3*s*. 6*d*. a Bushel; that the whole Mixture may be sold for 4*s*. 4*d*. the Bushel?

In this Case you must set down all the particular Rates, with the Mean Rate, and find their Difference just as before; without any Regard had to the Quantity given.

$$\text{Thus, Mean Rate } 52d.\left\{\begin{array}{l}\text{Wheat } 60d.\\ \text{Rye } 42d.\end{array}\right\}\left\{\begin{array}{l}10\\ 8\end{array}\right.$$

Then { *As the Quantity found by the Differences of the same Name with the Quantity given: Is to the Quantity given:: So is any of the other Quantities found by the Differences: To the Quantity of its Name.*

Thus 8 : 12 :: 10 : 15, the Quantity or Number of Bushels of Wheat required.

Quest. 5. How much Malaga at 7*s*. 6*d*. the Gallon, Sherry at 5*s*. the Gallon, and white Wine at 4*s*. 3*d*. the Gallon, must be mixed with 18 Gallons of Canary at 6*s*. 9*d*. the Gallon; that the whole Mixture may be sold for 6*s*. the Gallon?

The

The Terms being set down, *&c.* as before, will stand

$$\text{Thus, Mean Rate } 72d.\left\{\begin{array}{l}\left.\begin{array}{l}\text{Malaga } 90\,d.\\ \text{White } \;\;51\,d.\end{array}\right\}\left\{\begin{array}{l}21\\18\end{array}\right.\\ \left.\begin{array}{l}\text{Sherry } \;60\,d.\\ \text{Canary } 81\,d.\end{array}\right\}\left\{\begin{array}{l}9\\12\end{array}\right.\end{array}\right.$$

$$\text{Then, as } 12 : 18 :: \left\{\begin{array}{l}21 : 31\frac{1}{2} \text{ Gallons of Malaga.}\\ 18 : 27 \text{ Gallons of White.}\\ 9 : 13\frac{1}{2} \text{ Gallons of Sherry.}\end{array}\right.$$

That is, $31\frac{1}{2}$ Gallons of Malaga, 27 of white Wine, and $13\frac{1}{2}$ of Sherry, being mixed with 18 Gallons of Canary, will make the Mixture required.

$$\text{Or thus, } 72\left\{\begin{array}{l}\left.\begin{array}{l}\text{Malaga } 90\\ \text{Sherry } 60\end{array}\right\}\left\{\begin{array}{l}12\\18\end{array}\right.\\ \left.\begin{array}{l}\text{Canary } 81\\ \text{White } \;51\end{array}\right\}\left\{\begin{array}{l}21\\9\end{array}\right.\end{array}\right.$$

$$\text{Then, as } 21 : 18 :: \left\{\begin{array}{l}12 : 10\frac{6}{21} \text{ the Malaga.}\\ 18 : 15\frac{9}{21} \text{ the Sherry.}\\ 9 : 7\frac{15}{21} \text{ the White.}\end{array}\right\}\text{\&c.}$$

$$\begin{array}{ccc} & \textit{Gallons.} & & \textit{Pence.}\\ \textit{Proof.} & \left\{\begin{array}{l}10\frac{6}{21} \text{ at } 90\,d.\\ 15\frac{9}{21} \text{ at } 90\,d.\\ 7\frac{15}{21} \text{ at } 51\,d.\\ 18 \text{ at } 81\,d.\end{array}\right\} & \text{each} & \left\{\begin{array}{r}925\frac{15}{21}\\ 925\frac{15}{21}\\ 393\frac{9}{21}\\ 1458\end{array}\right.\\ & \text{Sum } 51\frac{9}{21} & & \text{Value} = 3702\frac{18}{21}\end{array}$$

Then $51\frac{9}{21}$) $3702\frac{18}{21}$ ($72d. = 6s.$ the Mean Rate.

Therefore the Quantities are as truly assigned here, as in the last Work.

Case III. The particular Rates of all the Ingredients proposed to be mixed; and the Sum of all their Quantities with the Mean Rate of the Sum being given; to find the particular Quantities of the Mixture.

This is called *Alligation Total*, and is thus performed.

Set down all the particular Rates, with the Mean Rate, and find their Differences, as before; add together all the Differences into one Sum:

Then { *As the Sum of all the Differences : Is to the Sum of all the Quantities given : : So is every particular Difference : To its particular Quantity.*

Quest. 6. Let it be required to mix Wheat at 5*s.* the Bushel, with Rye at 3*s.* 6*d.* the Bushel; so that the whole Quantity may be 27 Bushels, to be sold for 4*s.* 4*d.* a Bushel; what Quantity of each must be taken to make up the Mixture?

$$\text{Mean Rate } 52\left\{\begin{array}{ll}\text{Wheat} & 60d.\\ \text{Rye} & 42d.\end{array}\right\}\left\{\begin{array}{r}10\\ 8\end{array}\right.$$

$$\overline{18} \text{ their Sum.}$$

$$\text{Then } 18 : 27 :: \left\{\begin{array}{r}10 : 15\\ 8 : 12\end{array}\right\} \text{the Quantities required.}$$

Quest. 7. Suppose it were required to mix Malaga at 7*s.* 6*d.* the Gallon, with Canary at 6*s.* 9*d.* the Gallon, Sherry at 5*s.* the Gallon; and white Wine at 4*s.* 3*d.* the Gallon; so that the whole Mixture may be 90 Gallons; to be sold for 6*s.* the Gallon: How much of each Sort will compose that Mixture?

$$\text{Mean Rate} = 72d.\left\{\begin{array}{lr}\text{Malaga} & 90\\ \text{White} & 51\\ \text{Canary} & 81\\ \text{Sherry} & 60\end{array}\right.\begin{array}{l}\left.\begin{array}{l}\\ \\ \end{array}\right\}\left\{\begin{array}{r}21\\ 18\end{array}\right.\\ \left.\begin{array}{l}\\ \\ \end{array}\right\}\left\{\begin{array}{r}9\\ 12\end{array}\right.\end{array}$$

$$\overline{60} = \text{their Sum}$$

$$\text{Then } 60 : 90 :: \left\{\begin{array}{r}21 : 31\frac{1}{2}\\ 18 : 27\\ 9 : 13\frac{1}{2}\\ 12 : 18\end{array}\right\} \text{the Gallons of} \left\{\begin{array}{l}\text{Malaga.}\\ \text{White Wine.}\\ \text{Sherry.}\\ \text{Canary.}\end{array}\right.$$

$$\text{Or thus, } 72\left\{\begin{array}{lr}\text{Malaga} & 90\\ \text{Sherry} & 60\\ \text{Canary} & 81\\ \text{White} & 51\end{array}\right.\begin{array}{l}\left.\begin{array}{l}\\ \\ \end{array}\right\}\left\{\begin{array}{r}12\\ 18\end{array}\right.\\ \left.\begin{array}{l}\\ \\ \end{array}\right\}\left\{\begin{array}{r}21\\ 9\end{array}\right.\end{array}$$

$$\overline{60} \text{ their Sum.}$$

$$\text{Then } 60 : 90 :: \left\{\begin{array}{r}12 : 18\\ 18 : 27\\ 21 : 31\frac{1}{2}\\ 9 : 13\frac{1}{2}\end{array}\right\} \text{Gallons of} \left\{\begin{array}{l}\text{Malaga.}\\ \text{Sherry.}\\ \text{Canary.}\\ \text{White Wine.}\end{array}\right.$$

Either of these Ways do equally answer the Question, as may be easily tried by *Alligation Medial.* As before, &c.

Note, *The Work of these Proportions may be much shortened (especially when there are many Ingredients to be mixed) if you observe the same Method as was proposed in the* Rule of Fellowship, *page* 99, &c.

I have made use of the very same *Examples* both in *Alligation Medial*, and *Alternate*, throughout the three *Cases*; being, as I presume, much better than if they had been different ones; because the Learner may (if he consider them a little) easily perceive not only the Difference between the two Rules, but also wherein the

the chief Difference of each Cafe in the *Alternate Rule* depends, &c. Not but that I could have inferted many various *Examples*, as alfo the Manner of compofing Medicines, &c. which, for Brevity fake, I have omitted, and refer thofe that defire to fee into that Bufinefs to Sir *Jonas More's Arithmetick*, wherein he will find it largely handled. And as I fhall conclude with *Alligation Alternate*, which altho' it gives true Anfwers to Queftions of that Kind, with fome little Variety, according as the Ingredients are more or lefs in Number, as appears by the foregoing *Examples*; yet it will not give all the Anfwers fuch Queftions are capable of, nor perhaps thofe which fuit beft with the prefent Occafion: Nor can this Imperfection be remedied by common *Arithmetick*; but by an *Algebraic* Way of arguing it may; whereby all the poffible Anfwers to any Queftion may be clearly and eafily difcovered; as fhall be fhewed further on in the Second Part.

CHAP. X.

Of METALS *and their* SPECIFICK GRAVITIES, &c.

Sect. 1. *Of* GOLD *and* SILVER.

PURE Gold, free from Mixture with other Metals, ufually called Fine Gold, is of fuch a Nature and Purity that it will endure the Fire without wafting, although it be kept continually melted: And therefore fome of the antient Philofophers have fuppofed the Sun to be a Globe of liquid or melted Gold.

Silver having not the Purity of Gold, will not endure the Fire like it: Yet Fine Silver will wafte but very little by being in the Fire any reafonable Time; whereas Copper, Tin, Lead, &c. will not only wafte, but may be calcined or burnt to a Powder.

Both Gold and Silver in their Purity, are fo very flexible or foft (like new Lead, &c.) that they are not fo ufeful either in Coin, or otherwife (except to beat in Leaf-Gold or Silver) as when they are allayed, or mixed and hardened with Copper or Brafs. And although moft Places differ more or lefs in the Quantity of fuch Allay, yet in *England* it is generally agreed on, that,

Standard

Standard for Gold.

22 Carracts of Fine Gold, and 2 Carracts of Copper, being melted together ſhall be eſteemed the true Standard for Gold Coin, &c. *(The French and Spaniſh Gold being very near of the ſame Standard)* That is, if any Quantity or Weight of Fine Gold, be divided into Twenty-four equal Parts, and 22 of thoſe Parts be mixed with 2 of the like Parts of Copper; that Mixture is called Standard Gold.

Whence you may obſerve, that a Carract is not any certain Quantity or Weight, but $\frac{1}{24}$ Part of any Quantity or Weight; and the *Minters* and *Goldſmiths* divide it into 4 equal Parts, which they call Grains of a Carract; alſo they ſubdivide one of thoſe Grains, into Halves, Quarters, &c.

Standard for Silver.

Eleven Ounces and two Penny-weight of Fine Silver, and Eighteen Penny-weight of Copper being melted together, is eſteemed the true Standard for Silver Coin, called Sterling Silver. And ſo in Proportion for a greater or leſſer Quantity; which is a leſs Proportion of Allay for Silver, than the other is for Gold.

Note, When either Silver or Gold is finer than Standard, it is called Better; if coarſer, it is called Worſe; and that Betterneſs or Worſeneſs, is reckoned by Carracts and Grains of a Carract in Gold, and by Penny-weights in Silver; and is thus diſcovered: The *Goldſmiths* or *Refiners*, &c. take a ſmall Quantity of ſuch Gold as they intend to try (which they call making an *Aſſay)* and weigh it very exactly, then they put it into a Crucible, and melt it in a ſtrong Fire, ſo long, that if there be any Copper, or other Allay mixed with it, that Allay may be conſumed or burnt away: When it is cold they weigh it very exactly again, and if it have loſt nothing of its firſt Weight, they conclude it is Fine Gold, but if the Loſs be $\frac{1}{24}$ Part, they call it 23 Carracts Fine, or one Carract better than Standard: If it have loſt $\frac{2}{24}$ Parts it is 22 Carracts fine, or Standard: If $\frac{3}{24}$ Parts, it is ſaid to be 21 Carracts fine, or rather one Carract worſe than Standard, and ſo in Proportion as it happens to be better or worſe.

In the ſame Manner they make their Aſſay on Silver, only they compute its Loſs by Penny-weights, &c.

The Author of the *Preſent State of England*, mentioned before *(page* 32.) ſays,

' That

'That the *English* Coin may want neither the Purity nor 'Weight required, it is most wisely and carefully provided, 'that once every Year the chief Officers of the *Mint* appear be-'fore the Lords of the Council in the *Star-Chamber* at *Westmin-'ster*, with some Pices of all Sorts of Monies coined the fore-'going Year, taken at adventure out of the Mint, and kept un-'der several Locks, by several Persons, till that Appearance, and 'then by a Jury of 24 able *Goldsmiths*, in the Presence of the 'said Lords, every Piece is most exactly weighed and Assayed.'

This if it were constantly practised would keep our Coin to its true Standard, *&c.*

Many pretty Questions may be started concerning the Fineness of Gold and Silver, *&c.*

EXAMPLE 1.

If an Ingot of Silver weighing 787 *Oz.* 14 *Pwt.* 6 *Grains*, be 11 *Oz.* 6 *Pwt.* fine; How much fine Silver is there in it, and what amounts it to, at 5*s.* 1½*d.* the Ounce?

This Ingot is better than Standard by 4 *Pwt.* For 11 *Oz.* 2 *Pwt.* = 222 *Pwt.* the fine Silver in 12 *Oz.* of Standard. But 11 *Oz.* 6 *Pwt.* = 226 *Pwt.* the fine Silver in 12 *Oz.* according to the Question.

First 787 *Oz.* 14 *Pwt.* 6 *Grains* = 378102 *Grains.*

And 12 *Oz.* = 240 *Pwt.*

Then, As 240 : 226 : : 378102 : 356046 $\frac{1}{20}$ = 741 *Oz.* 15 *Pwt.* 6 $\frac{1}{20}$ *Grains* the fine Silver in that Ingot.

Which at 5*s.* 1½*d.* the Ounce, amounts to 190*l.* 1*s.* 6*d.* and near a Half-penny.

EXAMPLE 2.

If an Ingot of Gold weighing 115 *Oz.* 13 *Pwt.* 18 *Grains*; be ¼ of a Grain worse than Standard: How much Standard Gold is there in it, and what comes it to at 3*l.* 11*s.* an Ounce?

First 115 *Oz.* 13 *Pwt.* 18 *Grains* = 55532 Grains *Troy.*

Then 24) 55530 (2313,75 = a Carract of that Quantity.

And 4) 2313,75 (578,4375 = a Grain of that Carract.

Consequently 4) 578,4375 (144,609375 = ¼ of a Grain.

Again, 2313,75×22=50902,5 ought to be the fine Gold in that Ingot, if it had been Standard.

But

But 50902,5—144,609375=50757,890625 is the Quantity of fine Gold according to the Question.

Therefore 60902,5 : 50757,890625 : : 55530 : 55372,244 &c. Grains = 115 *Oz.* 7 *Pwt.* 4,244 &c. Grains *Troy*, being the Quantity of Standard Gold in that Ingot, as was required.

Next for the Value of it at 3*l.* 11*s.* *per* Ounce; 1 *Oz.*=480 Grains; and 3*l.* 11*s.* = 71*s.*

Consequently 480 : 71 : : 55372,244 &c. : 8190,4777 &c. =409*l.* 10*s.* 5 $\frac{3}{4}$*d.* very near; being the Value of that Ingot, as was required.

Or the last Question may be otherwise wrought thus; 115 *Oz.* 13 *Pwt.* 18 *Grains* = 115,6875. And $\frac{1}{4}$ or a Grain of a Carract is $\frac{1}{16}$ (*viz.* the $\frac{1}{4}$ of $\frac{1}{4}$) Then 22—$\frac{1}{16}$=21$\frac{15}{16}$=21,9375.

Consequently 22 : 21,9375 : : 115,6875 : 115,358842 &c. =115 *Oz.* 7 *Pwt.* 4,244 *Grains*, &c. as before.

Next for the Value; as 1 : 3,55 : : 115,358842 : 409,523889 =409*l.* 10*s.* 5 $\frac{3}{4}$*d.* very near: as before.

Sect. 2. *The* Specifick Gravity *of* Metals, &c.

I Take an Enquiry made about the different Gravities, or Weights of Metals, and other Bodies, to be (not only a Work of Curiosity, but also) of very good Use upon many Occasions. Therefore several Authors have given us such Proportions, or Difference of their Weights, as they are said to have one to another; supposing every one of them to be of the same Magnitude or Bigness. Some of which I shall here insert.

1. *Henry Van Etten*, in his *Mathematical Recreations*, printed *Anno* 1633, sets down the Proportion of their Weights thus; Gold 1875 . Lead 1165 . Silver 1040 . Copper 910 . Iron 810: Tin 150 . Water 100.

2. One *Alsted*, in his *Encyclopædia*, printed 1649, hath them thus: Gold 1875 . Quickfilver 1500 . Lead 1165 . Silver 1040 . Copper 910 . Iron 806. Tin 750 . Honey 150 . Water 100 . Oil 90. These seem to be taken from those of *Van Etten*'s, with some Additions only.

3. The ingenious Mr. *Oughtred*, in his *Circles of Proportions*, printed *Anno* 1660, hath their Proportions (according to the Experiments of one *Marinus Ghetaldi*, in his Tract called *Archimedes Promotus*) thus: Gold 3990 . Quickfilver 2850 . Lead 2415 . Silver 2170 . Brass 1890 . Iron 1680 . Tin 1554.

4. In

4. In the Philosophical Transactions, *(Number* 169 and 199) there is an Account of a great many Experiments of this Kind; from whence I collected these following, *viz.* Gold 17888. Mercury 14019. Lead 11345. Silver 10535. Copper 8843. Hammered Brass 8349. Cast Brass 8100. Steel 7852. Iron 7643. Tin 7321. Pump-water 1000.

These last Proportions being approved of and published by Order of the *Royal Society*, seem to be unquestionably true: Nevertheless, because they differ so much from the beforementioned *(and those from one another)* I have for my own Satisfaction made several Experiments of that Kind: And have *(I presume)* obtained the Proportions of Weight that one Body bears to another of the same Bulk or Magnitude, as nicely as the Nature of such Matter, which may be contracted or brought into a lesser Body *(viz.* either by drying, or hammering, or otherwise) will admit of; which are as followeth:

A Cubick Inch of		Ounces Troy.	Ounces Averd.
	Fine Gold, is	10.359273=	11,365602
	Standard Gold,	9,962625=	10,930422
	Quickfilver,	7,384411=	8,101753
	Lead,	5,984010=	6,553885
	Fine Silver,	5,850035=	6,418324
	Standard Silver,	5,556769=	6,096569
	Rose Copper,	4,747121=	5,208369
	Plate Brass,	4,404273=	4,832116
	Cast Brass,	4,272409=	4,6[illegible]300
	Steel,	4,142127=	4,544505
	Common Iron,	4,031361=	4,422979
	Block Tin,	3,861519=	4,236638
	Fine Marble,	1,429411=	1,568859
	Common Glass,	1,360841=	1,493037
	Alabaster,	0,988456=	1,084477
	Dry Ivory,	0,962083=	1,055542
	Dry Box-wood	0,543282=	0,596057
	Sea Water,	0,542742=	0,594894
	Common clear Water,	0,527458=	0,578697
	Red Wine,	0,523766=	0,574646
	Proof Spirits or Brandy,	0,489268=	0,536796
	Sound dry Oak,	0,489008=	0,536569
	Linseed Oil,	0,491591=	0,539345
	Oil Olive,	0,481569=	0,528350

In this Table you have the ſpecifick Gravity or Weight of a Cubick Inch, of various Sorts of Bodies, both in *Troy* Ounces and *Averdupois* Ounces, and decimal Parts of an Ounce, which I can aſſure you, required more Charge, Care, and Trouble, to find out nicely, than I was at firſt aware of.

Now from hence it will be eaſy to determine the Weight of any propoſed Quantity, of the ſame Matter and Kind with thoſe in the Table; its ſolid Content being given in Cubick Inches. For it is plain, that if the Number of Cubick Inches contained in any given Quantity, be multiplied with the tabular Weight of one Inch, *(of the ſame Kind of Matter)* the Product will be the Weight of that Quantity in Ounces, *&c.*

EXAMPLE.

Suppoſe it were required to find the Weight of a Piece of Marble, containing three ſolid Feet, and 40 Cubick Inches.

Firſt $1728 \times 3 = 5184$ the Cubick Inches in 3 ſolid Feet.
And $5184 + 40 = 5224$ the Number of Cubick Inches in the Piece of Marble.
Then $5224 \times 1{,}429411 = 7410{,}066624$ Ounces *Troy*.
Or $5224 \times 1{,}568859 = 8195{,}719416$ Ounces *Averdupois*.

The Weight of that Piece of Marble, in Ounces, *&c.* which is eaſily brought into Pounds, *&c.* The like for any of the reſt.

The Converſe of this Work is as eaſy; *viz.* if the Weight of any propoſed Quantity be given, thence to find the ſolid Content of that Quantity in Cubick Inches, *&c.*

Thus, divide the given Weight of the propoſed Quantity *(it being firſt reduced into Ounces,* &c.) by the tabular Weight of one Inch *(of the ſame Kind of Matter)* and the Quotient will be the Number of Cubick Inches contain'd in that Quantity.

Note, If you would find what Weight any Quantity of thoſe Bodies mentioned in the Table will have, when it is immerſed or put into Water, you muſt ſubſtract the Weight of an equal Quantity of Water (with that of the Body) from the Weight of the propoſed Body (if it be heavier than Water) and there will remain the Weight required. As for Inſtance,

$$\left.\begin{array}{l}\text{A Cubick Inch of Lead}=5.984010\\ \text{A Cubick Inch of Water}=0{,}542742\end{array}\right\}\text{Ounces } \textit{Troy}, \text{ \&c.}$$

their Difference is, $=5{,}441268$ the Weight of a Cubick Inch of Lead in the Water, *&c.*

CHAP.

CHAP. XI.

Evolution, *or Extracting the* Roots *out of all* Single Powers, *by one Geometrical Method.*

SECT. I.

EVolution is the unravelling, or as it were the unfolding and resolving any proposed Power or Number, into the same Parts of which it was composed, or supposed to be made up. Now in order to perform that, it will be convenient to consider how those Powers are composed, &c.

A Square Number is that which is equally equal; or which is contained under two equal Numbers. *Euclid* 7. *Def.* 18. Thus the square Number 4, is composed of the two equal Numbers 2 and 2, *viz.* $2\times2=4$. Or the square Number 9, is composed of the two equal Numbers 3 and 3, *viz.* $3\times3=9$: according to *Euclid.* That is, if any Number be multiplied into itself, that Product is called a square Number.

A Cube is that Number, which is equally equally equal, or which is contained under three equal Numbers. *Eu.* 7. *Def.* 19. Thus the Cube Number 8 is composed of the three equal Numbers 2 and 2 and 2, *viz.* $2\times2\times2=8$, &c. That is, if any Number be multiplied into itself, and that Product be multiplied with the same Number; the second Product is called a Cube Number.

These two, *viz.* the Square and Cube Numbers, borrow their Names from *geometrical* Extensions or Figures; as from the three signal Quantities mentioned in *page* 2. That is, a *Root* is represented by a Line or Side, having but one Dimension, *viz.* that of Length only. The Square is a Plane or Figure of two Dimensions, having equal Length and Breadth. The Cube is a solid Body of 3 Dimensions, having equal Length, Breadth, and Thickness: But beyond these three, Nature proceeds not, as to *local* Extension. That is, the Nature of Place or Space, admits no Room for other Ways of Extension than Length, Breadth and Thickness. Neither is it possible to form, or compose any Figure or Body beyond that of a Solid.

And therefore all the superior Powers above the Cube or third Power; as the *Biquadrat* or fourth Power, the *Sursolid* or fifth Power, &c. are best explained and understood by a Rank or Series of Numbers in *geometrical Proportion.* For Instance: Suppose any Rank of *geometrical Proportionals,* whose first Term and Ratio are the same; and to them let there be assigned a Series of Numbers in *arithmetical Progression,* beginning with an Unit or 1, whose common Difference is also 1, as in *page* 79.

 Thus,

Thus, $\left\}\begin{array}{l}1 \;.\; 2 \;.\; 3 \;.\; 4 \;.\; 5 \;.\; 6 \;.\; 7 \text{ Indices.}\\ 2 \;.\; 4 \;.\; 8 \;.\; 16 \;.\; 32 \;.\; 64 \;.\; 128 \text{ \&c. in } ∺\end{array}\right.$

Then are thoſe Numbers in ∺ produced by a continual Multiplication of the firſt Term or Root into itſelf; and thoſe in *arithmetical Progreſſion* or INDICES, do ſhew what Degree or Power each Term in the *geometrical Proportion* is of. For Example; In this Series of ∺ 2 is both the firſt Term or Root, and common Ratio of the Series. Then 2×2=4 the ſecond Term or Square; and 2×2×2=8, or 4×2=8, the Cube or third Term; 2×2×2×2=16, or 8×2=16 the fourth Term or Biquadrat. And ſo on for the reſt.

Note, *This is called* INVOLUTION, viz. *When any Number is drawn into itſelf, and afterwards into that Product,* &c. *it is ſaid to be ſo often involved into itſelf; and the Indices are the Exponents of their reſpective Powers ſo involved.*

And according to the Involutions, is formed the following Table of Powers; wherein the Root is only one ſingle Figure.

Root, or ſingle Side.	*Square or ſecond Power.*	*Cube, or the third Power.*	*Biquadrat, or Square ſquared; being the fourth Power.*	*Surſolid, or the 5th Power.*	*Square cubed, or Cube ſquared; the ſixth Power.*	*The ſecond Surſolid, or ſeventh Power.*	*The Biquadrat ſquared, or the eighth Power.*	*The Cube cubed, or the ninth Power, &c.*
	Index (2)	*Index* (3)	*Index* (4)	*Index* (5)	*Index* (6)	*Index* (7)	*Index* (8)	*Index* (9)
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

This Table plainly ſhews (by Inſpection) any Power (under the Tenth) of all the nine Figures; and from thence may be taken the

the nearest Root of any Square, Cube, Biquadrat, &c. of any Number whose Root or Side is a single Figure.

But if the Root consists of two, three, or more Places of Figures, then it must be found by piece-meal, or Figure after Figure, at several Operations.

The Extraction of all Roots, above the Square (*viz.* of the Cube, Biquadrat, Sursolid, &c.) hath heretofore been a very tedious and troublesome Piece of Work: All which is now very much shortened, and rendered easy, as will appear further on.

When any Number is proposed to have its Root extracted, the first Work is to prepare it, by Points set over (or under) their proper Figures; according as the given Power, whose Root is sought doth require; and that is done by considering the Index of the given Power, which for the Square is 2, for the Cube 3, for the Biqadrat is 4, &c. (as in the precedent Table); then allow so many Places of Figures in the given Power, for each single Figure of the Root, as its Index denotes; always beginning those Points over the Place of Unity, and ascend towards the Left-Hand, if the given Number be Integers, and descend towards the Right-Hand in decimal Parts. As in these following.

Suppose any given Number; as 75640387246 which I shall all along hereafter call the Resolvend.

Then if it be required to extract any of the following Roots, it must be pointed (according to the forementioned Consideration) in this Manner:

Viz. For the
- *Square Root* Thus 75640387246
- *Cube Root* 75640387246
- *Biquadrat Root* 75640387246
- *Sursolid Root* 75640387246

Or suppose the Number to be 0,6740359820

Then for the
- *Square Root* Thus 0,6740359820
- *Cube Root* 0,674035982
- *Biquadrat Root* 0,674035982000

Now the Reason of pointing the given Resolvend in this Manner, *viz.* the allowing two Figures in the Square, three Figures in the Cube, and four Figures in the Biquadrat, &c. For one Figure in the Root, may be made evident several Ways; but I think

think it is easily conceived from the Table of single Powers, wherein you may observe that all the Powers of the Figure 9 (which is but a single Figure) have the same Number of Places of Figures, as the Index of those Powers denotes: Therefore so many Places of Figures must be taken or assigned for every single Figure in the Root. Consequently by these Points is known how many Places of Figures there will be in the Root, *viz.* So many Points as there are, so many Figures there must be in the Root, and whether they must be Integers or decimal Parts, is easily determined by the respective Places of the Points.

Sect. 2. *To* Extract *the* Square Root.

AND first how to extract the Square Root, according to the common Method.

Having pointed the given Resolvend into Periods of two Figures as before directed; then by the Table of Powers (or otherwise) find the greatest Square that is contained in the first Period towards the Left-hand (setting down its Root, like a Quotient Figure in *Division*) and substract that Square out of the said Period of the Resolvend: To the Remainder bring down the next Period of Figures, for a Dividend, and double the Root of the first Square for a Divisor; enquiring how oft it may be had in that Dividend, so as when the Quotient Figure is annexed to the Divisor, and that increased Divisor multiplied with the same Quotient Figure, the Product may be the greatest Number that can be taken out of that Dividend; which substract from the said Dividend, and to the Remainder bring down the next Period of Figures, for another new Dividend: Then see how often the last increased Divisor, can be had in the new Dividend *(with the same Caution as before,* viz.) so as that the Quotient Figure being annexed to the Divisor, and that increased Divisor multiplied with the same Quotient Figure, their Product may be the greatest Number that can be substracted from the new Dividend. (As before.) And so proceed on from Period to Period *(viz.* from Point to Point) in the very same Manner, until all be finished.

An Example or two being well observed will render the Work of forming the new Divisors, &c. more plain and easy than can be expressed in a Multitude of Words.

Example 1. Let it be required to extract the Square Root out of 572199960721. This Resolvend being prepared or pointed as before directed, will stand

Thus,

Thus, 572199960721 (756439 the Root.
49=the greatest Square in 57.

1. Divisor	145)	821
	5	725=145×5
2. Divisor	1506)	9699
	6	9036=1506×6
3. Divisor	15124)	66396
	4	60496=15124×4
4. Divisor	151283)	590007
	3	453849=151283×3
5. Divisor	1512869)	13615821
	9	13615821=1512869×9

Proof 756439×756439=572199960721 the Resolvend.

Example 2. What is the Square Root of 1850701,764025?

Operation 1850701,764025 (1360,405

	1
23)	85
3	69
266)	1607
6	1596
17204)	1101,76
4	1088 16
1720805)	13 604025
5	13 604025
	(0)

Hence 1360,405 is the Root required.

Ex. 3. What is the Square Root of 0,06076225 Decimal Parts?

Operation 0,06076225 (0,2465 the Root required.

	,04=,2×,2
,44)	207
4	176
,486)	3162
6	2916
,4925)	24625
5	24625
	(0)

Proof 0,2465×0,2465= 0,06076225 the Resolvend.

What

What is here done in whole Numbers, mixed Numbers, and Decimals, may also be done in Vulgar Fractions; if you first change the given Fraction into Decimals. (As in *Sect.* 5. *p.* 68.

Example 4. Let it be required to extract the Square Root of $\frac{16}{25}$ First $\frac{16}{25}$=0,64

```
Then 0,64 (,8 the Root required.
      ,64
      ---
      (0)
```

In these four Examples the Resolvend hath been a perfect Square; and therefore the Root hath been extracted without leaving any Remainder: But it very often happens that the Resolvend is not a true figurate Number, according to the proposed Power. That is, it is not a perfect Square, Cube, Biquadrat, &c. and then something will remain after the Extraction hath been made throughout all the Points. Such Numbers are called Surd Numbers, and their Roots can never be truly found, but will become a continued Series, *ad infinitum:* If to the Remainder there be still annexed Cyphers according as the proposed Power requires, *viz.* by two's in the Square, three's in the Cube, four's in the Biquadrat, &c. And the Operations continued on as before.

Example 5. Suppose it were required to extract the Square Root of 6968

```
          Operation 6968 (83,4745, &c.
                    64
                    ---
163)                568
  3                 489
-----               ----
1664)               79,00
   4                66 56
------              -----
16687)              12 4400
    7               11 6809
-------             -------
166944)               759100
     4                667776
--------              -------
1669485)              9132400
      5               8347425
--------              -------
1669490)               784975 &c.
```

Then the Root of any Surd Number may be continued on to what Exactness you please, but cannot be truly found.

In my *Compendium of Algebra*, Chap. 9. I have proposed another Way of extracting the Square Root, and there given Examples of the Work; which to avoid Prolixity is thus;

Having

Having pointed the given Resolvend, and taken the greatest Square to the first Point from it, as before. Then divide the Remainder of the whole Resolvend by 2 (that is, half it) and point it a-new. (This I call a new Dividend) Then make the Root of the first Square a Divisor, inquiring how oft it may be found in the new Dividend to the next Figure forward, reserving that Figure under the next Point for the half Square of the Quotient Figure. Which being found, multiply the Divisor with it, adding to that Product the Tens of the half Square if there be any, as in plain Division. Then annex the Quotient Figure to the last Divisor for a new Divisor, with which proceed in all Respects as with the last Divisor; and so on until all be finished.

Example 6. What is the Square Root of 2990667969

Operation 2990667969
— 25 (5 The first single Root

2) 490667969 The Remainder to be divided by 2.

First Root 5) 245333984,5 (54687
+4 208 $=5\times4$: $+\frac{1}{2}$ the Square of 4, *viz.* $\frac{16}{2}=8$.

Divisor 54) 3733
+6 3258 $=54\times6$: $+\frac{1}{2}$ the Square of 6.

Divisor 546) 47539
+8 43712 $=546\times8$: $+\frac{1}{2}$ the Square of 8.

Divisor 5468) 382784,5
+7 382784,5 $=5468\times7$: $+\frac{1}{2}$ the Square of 7.
(0)

Hence the Root is found to be 54687, as was required.

All the Difficulty in this Method is only the true placing of the half Square of the Quotient Figure, when it happens to be an odd Number: In that Case you must bring down one Figure more of the Dividend; *viz.* of the next Period; under which, place the odd 5 that will always arise from the half Square of an odd Number: As 7 whose Square is 49; the Half of which is 24,5 to be placed as in the last Operation of this Example.

N. B. *When the Number of Figures in the Root of any Surd Number are limited; you need not proceed in extracting the whole Root as before; but only to one Figure more than half the designed Number of Figures; for the rest may be obtained by plain Division only.*

Example 7. Suppose it were required to extract the Square Root of 7 (a Surd Number) to have 12 Places of Figures in it.

	7 (2,645751	First Part of the Root.
	4	
Remainder	3	
2)	1,50=Half the Remainder.	
+,6	1,38=2×,6 : +½ the Square of 0,6=0,18	
2,6)	1200	
+,04	1048	
2,64)	152000	
+,005	132125	
2,645)	1987500	
+,0007	1851745	
2,6457)	13575500	
+,00005	13228625	
2,64575)	34687500	
+,000001	26457505	
2,645751	8229995	

Having thus got 7 of the 12 Figures required in the Root; the rest may be easily found by the contract Way of Division proposed in *page* 68.

Thus 2,645751)	8229995	(2,64575131106
....	7937253	
	292742	
	264575	
	28167	
	26457	
	1710	
	1097	
	(13)	

Hence I find the Root of 7 to be 2,64575131106, *as was required.*

Thus you have two Ways of extracting the Square Root, either of which may be practised as every one likes best.

Sect.

Sect 3. *To* Extract *the* Cube Root.

THE Method I shall here propose for extracting the Cube Root admits of two Cases; both which are to be very well observed.

Having pointed the given Resolvend, (as before directed) *viz.* into Periods of three Figures; then seek a Cube Number by the Table of Powers (or otherwise) that comes nearest to the first Period of the Resolvend, whether it be greater or less than that Period.

Case 1. If the Cube Number so taken, be less than the first Period of the Resolvend, call its Root Less than Just: And substract that Cube from the first Period of the Resolvend.

Case 2. But if that Cube be greater than the first Period of the Resolvend, call its Root More than Just: And substract the Resolvend from that Cube, annexing Cyphers to it, that so Substraction may be made.

To the first Root, whether it be less or more than Just, annex so many Cyphers as there are remaining Points over the whole Numbers of the Resolvend, and multiply it with 3: Then making that Product a Divisor, by which you must divide the Difference between the Resolvend and the foresaid Cube; that Quotient will be the Resolvend depressed to a Square, and therefore must be pointed as such, *viz.* into Periods of two Figures each. That being done, make the first Root (without those Cyphers that were annexed to it) a Divisor, inquiring how oft it may be found in the first Period of the new Resolvend (as before in extracting the Square Root) with this Consideration, that if the Root (now a Divisor) be less than Just, as in *Case* 1. you must annex the Quotient Figure to it, and then multiply the Root so increased, into the said Quotient Figure; setting down the Unit's Place of their Product under the pointed Figure of that Period, substracting it, as in Division. And so on from one Period to another, as before.

But if the said Root (now a Divisor) be more than Just, as in *Case* 2. Then you must substract the Quotient Figure from a Cypher annexed, or supposed to be annexed, to the said Divisor; multiplying the Root so decreased into the Quotient Figure; setting down their Product as before, &c. An *Example* or two in each *Case* will render the Work plain and easy.

Note, *Each Quotient Figure ought always to be twice added to the Divisor, if the tabular Cube was taken less than Just, or twice substracted from it, if greater,* viz. *once before you multiply by it, and once with the next Quotient Figure: as will be shewn in the following Examples; which are therefore more exact and concise than as done by the Author in all the former Editions of his Work.*

 Ex. 1.

Ex. 1. What is the Cube Root of 146363183 the given Resolvend,

to be pointed thus 146363183 (5 the first Root, less than Just.
125=the nearest Cube to 146

500×3=1500) 21363183 (14242,12 new Resolvend
First Root 5
+2

1 Divisor 52) 14242,12 (527 the Root required.
+7 104

2 Divisor 527) 3842
3689

153 the Remainder to be rejected.

Here the Root 527 is the true Root at the first Operation, as may be easily tried by involving it.

That is 527×527×527=146363183 the given Resolvend. But if it had not been the true Root, then every Thing that hath been here done must have been repeated; only instead of the first single Root (*viz.* 5.) you must have taken the increased Root (*viz.* 527) and this I call a second Operation; which would increase the last Root to nine Places of Figures, *viz.* every Operation triples the Number of Places in the last Root; as will appear further on.

N. B. *It often happens that four, five, and sometimes more Places of Figures may be taken into the Root: especially when the second Place proves to be a Cypher. That is, when the first Cube comes very near to the first Period of the Resolvend.*

EXAMPLE 2.

What is the Cube Root of 67507824239 (4000 Root less than Just.
First nearest Cube=64

Root 4000×3=12000) 3507824239 (292318,68
First Root 4
+07

1 Divisor 407) 292318,68 (4071,79
+071 2849

2 Divisor 4141) 7418
+ 1,7 4141

3 Divisor 4142,7) 3277,68
+ ,79 2899,87

4 Divisor 4143,49) 377,79 &c.

In

In this *Example* I have taken six Figures into the Roots, because the second Place proved to be a Cypher. And in these six the Excess is not an Unit in the last Place; for if there were made a second Operation, the Root would be 4071,78 *&c.* as may be easily tried.

EXAMPLE 3.

Let it be required to extract the Cube Root out of this Number,

Viz. 976379602989073960279630298890
The nearest Cube to 976 is 1000 whose Root is 10 more than Just,
its Cube 1000000000000000000000000000000
— 976379602989073960279630298890 the Resolvend

Remains 23620397010926039720369701110
The first Root 10000000000 × 3 = 30000000000 the Divisor.
Then 30000000000) 23620397010926039720369701110 (787346567030867990 for a new Resolvend.

1st Root	10		
	— 007		
1 Div.	993)	787346567030867990	10000000000 = 1st Root.
	— 79	6951	0079364 &c. substract.
2 Div.	9851)	92246	Remains 9920636000 the Root true to the 6th Figure, and only too little by an Unit at the 7th, at the first Operation.
	— 93	88659	
3 Div.	98417)	358756	
	— 36	295251	
4 Div.	984134)	6350570	
	— 64	5904804	
5 Div.	9841276) &c.	44576630 &c.	

For a second Operation (if you require no more than ten Places of Figures true in the Root) you need only assume 9920000000; which being less than just, proceed with as follows:

From the given Resolvend = 976379602989073960279630298890
Sub. Cube of 9920000000 = 976191488000000000000000000000
Remainder 188114989073 96 &c.

Then 3 × 992 &c. = 2976 &c.) 18811498907396 &c. (6321068181 &c. for a new Resolvend.

99200		
+ 06		
	. . . . .	9920000000 the Root assumed
99206)	6321068181	637163,5 add
+ 63	595236	9920637163,5 the Root true to the tenth Figure, and only too much by an Unit in the eleventh.
992123)	3687081	
+ 37	2976369	
9921267)	71071281	
+ 7 &c.	69448869	
992127)	1622412	
. . . .	992127	* Here the Additions of the Quotient Figure being of no Consequence, therefore the Division is carried on from hence, as in *page* 68.
	630285	
	595276	
	35009	
	29763	
	5246	
	4960	
	&c.	

In the same Manner the Cube Roots of Decimal Parts; or of Vulgar Fractions, being first changed into Decimals, may be extracted.

Sect. 4. *To* Extract *the* Biquadrat Root.

IN extracting the Biquadrat Root, or that of the fourth Power; (and indeed the Roots of all even Powers) there are some small Difficulties, not so easily expressed and explained in a few Words, as they are by an *Algebraic Theorem* (such as shall be shewed further on) I have therefore in this Place made choice of extracting such Roots by two several Extractions; and the rather, because I presume the Reader by this Time thoroughly acquainted with the Business of extracting the Square Root, by which this may easily be performed. Thus:

First, Extract the Square Root of the proposed Resolvend, then the Square Root of that first Root will be the Biquadrat Root required.

Example 1. What is the Biquadrat Root of 4857532416? First extract its Square Root,

Thus

Thus 4857532417
— 36 = the greatest Square, whose Root is 6.

1257532416 Remainder to be divided by 2.

First Root 6) 628766208 (69696
+ 9 5805

69 4826
+ 6 4158

696 668620
+ 9 626805

6969 418158
418158
(0)

Then 69696 { being the first Root, whose Square Root must now be extracted.
— 4

29696 Remainder to be divided by 2.

First Root 2) 14848 (264 the Biquadrat Root as was required.
+ 6 138

26) 1048
+ 4 1048

264 (0)

This is so easy I need not insert any more Examples.

Sect. 5. *To* Extract *the* Sursolid Root.

Having pointed the given Resolvend according as its Index denotes, *viz.* into Periods of five Figures; seeking such a Sursolid Number in the Table of Powers (or otherwise) as comes the nearest to the first Period of the Resolvend, whether greater or less; and call its respective Root accordingly, *viz.* more than Just, or less than Just; annexing so many Cyphers to it, as there are remaining Periods of whole Numbers in the Resolvend; as before in extracting the Cube Root: Then find the Difference between the Resolvend, and the Sursolid Number so taken, by substracting the lesser from the greater (as before in the Cube). Next find the Cube of the aforesaid Sursolid Root with its annexed Cyphers (which you may also do by the Table of Powers) and multiply that Cube with 5 the Index of the Sursolid, the Product must be a Divisor, by which the Difference between the Resolvend and the Sursolid Number must be divided; that so

so it may be depressed to a Square, (as before in the Cube) which must be pointed into Periods of two Figures each, calling it the new Resolvend (as before). Then make the first Root, without its Cyphers, a Divisor, enquiring how oft it may be found in the first Period of the new Resolvend, with this Consideration, if the Root (now a Divisor) be less than Just, you must annex twice the Quotient Figure to it; but if it be more than Just, you must substract twice the Quotient Figure from a Cypher either annexed, or supposed to be annexed to that Divisor or Root, multiplying it so increased or diminished, with the said Quotient Figure, setting down their Product, &c. as before. An *Example* in each *Case* will render it plain and easy.

Example 1. Suppose it be required to extract the Sursolid Root out of this Number 12309502009375.

12309502009375 The Resolvend pointed.

The nearest Sursolid Number to 1230, the first Period of the Resolvend, is 1024, whose Root is 4 being less than Just.

Therefore 12309502009375
— 1024

2069502009375 their Difference.

Next the Cube of 400 is 64000000 per Table, &c.

And 64000000×5=320000000 the Divisor.

Then 320000000) 2069502009375 (6497 &c.

First Root	400			
+ 2×10=+	20		400	
1 Divisor	420	6467	+ 15	
+ 2×5=+	10	42	415	Root true
	430	2267		
		2150		
		117		the Remainder to be rejected.

That is 415 is the Sursolid Root of the given Resolvend. As may be easily tried by involving it to the fifth Power. *Viz.* $415 \times 415 \times 415 \times 415 \times 415 = 12309502009375$ the given Resolvend.

Note, *Here again the double Quotient Figure ought to be twice added or substracted, in the same Manner as the single one was directed for the Cube Root*, page 131, *and the Operation for the Sursolid Root in these two Examples is performed accordingly; contrary to what was heretofore done by the Author.*

Example

Example 2. What is the Surfolid Root of 232783459873
The nearest Surfolid Number to 232 is 243 whose Root is 3 being more than just.

Therefore 243000000000
—232783459873

Remains 10216540127 For a Dividend.

The Cube of 300 is 27000000 and 27000000×5=135000000
Then 135000000) 102165440127 (756,7810 new Resolvend.

First Root	300		
—2×2=	—4		
1 Divisor	296)	756,7810	300 —2,566
—4—2×0,5=	—5,0	592	297,434 The Root only too little by 2 in the lowest Figure.
2 Divisor	291,0)	164,78	
—1—2×0,06=	—1,12	145,50	
3 Divisor	289,88) &c.	19,2810 &c.	

Now the Reason why this Root comes out to so many Places of Figures at the first Operation, is because the first Surfolid Number was so near the Resolvend, &c. As before.

Sect 6. *To* Extract *the* Root *of the* Square cubed.

THIS may be easily performed by two Extractions, according as its Name denotes. Thus, first extract the Square Root of the given Resolvend; then extract the Cube Root of that Square Root, and it will be the Root required: That is, it will be the Root of the sixth Power. Or thus, first extract the Cube Root of the Resolvend; then extract the Square Root of that Cube Root, and it will be the Root required.

EXAMPLE 1.

Let it be required to extract the Square cubed Root out of this Number 1452205373535515625 the Resolvend.

First I extract the Square Root of this Resolvend, which I take to be the best and easiest Way.

```
Thus         145220537353515625
            — 9
Remains      55220537353515625 to be halfed.

Then  3)        27610268676757812,5 (381078125
     + 8        272

      38)        4102
     +  10       3805

      3810)      2976867
     +     7     2667245

      38107)      3096226
     +      8     3048592

      381078)      47634757
     +       1     38107805

      3810781)      95269528
     +        2     76215622

      38107812)     1905390612,5
     +         5    1905390612,5
      381078125         (0)
```

Having found the Square Root of the given Resolvend, I proceed to extract the Cube Root of that Square Root.

```
        That is, of  381078125
                   — 343=the nearest Cube, its Root is 700
Then 700×3=2100) 38078125 (18161

First Root  7 . .
          + 2
                                   ( 700
1 Divisor   72.)      18161        ( + 25
          +  25       144             725
2 Divisor   745)      3761
                      3725
                      (36)
```

Hence I find 725 to be the Square Cube Root required; as may easily be tried by involving it to the sixth Power. That is, 725×725×725×725×725×725 will be found=14522053-7353515625 the given Resolvend.

Sect.

Sect. 7. *To* EXTRACT *the* ROOT *of the* seventh POWER.

HAving pointed the given Resolvend, as its Index denotes, *viz.* into Periods of seven Figures, seek out such a Number of the seventh Power, by the Table of Powers, as comes nearest to the first Period of the Resolvend; whether it be greater or lesser, calling its respective Root more than Just, or less than Just, annexing its proper Number of Cyphers, &c. as in the Cube and Sursolid.

Then find the Difference between the given Resolvend, and that Number of the seventh Power (found by the Table of Powers) by substracting the lesser from the greater.

Next find the Sursolid or fifth Power of that Root with its annexed Cyphers (which you may also do by the Table of Powers) and multiply that Sursolid Number with 7, the Index of the given Resolvend; that Product must be a Divisor, by which the foresaid Difference must be divided, that so it may be depressed to a Square, to be pointed, &c. as before in the Cube, &c. then make the first Root, without its Cyphers, a Divisor; working with it and the new Resolvend (as before) only here you must increase, or diminish the Divisor with thrice the Quotient Figure *.

Example. What is the second Sursolid Root, or that of the seventh Power,

of 38298655395507812 5 the Resolvend pointed.
— 2187 the nearest of the seventh Power.

16428655395507812 5 their Difference.

The first Root is 300 being less than Just, and the fifth Power of 300 is 2430000000000 which being multiplied with 7 is 17010000000000 for a Divisor, by which the aforesaid Difference must be divided; which contracted may stand thus, 1701) 16428655 (9658,23 &c.

First Root 300
$+3\times 20=+60$

1 Divisor 360) 9658 (300
$60+3\times 05=+75$ 72 +25
2 Divisor 435) 2458 325=the true Root required.
2175
283 the Remainder to be rejected, as [before.

* That is, by twice adding or substracting the triple Quotient Figure, as was done with the double Quotient Figure for the Root of the fifth Power, page 136; and the single Quotient Figure for the Cube Root, page 131.

Hence I have found 325 to be the true Root required, that is the true Root of the ſeventh Power.

I think it needleſs to proceed farther, *viz.* to inſert *Examples* of higher Powers. For if what is already done be well underſtood, it will be eaſy to conceive how to proceed in extracting the Root of any ſingle Power how high ſoever it be (for the Method is general and alike in all Powers) due Regard being had to their Indices; and to the firſt ſingle Side or Root. That is, whether it be More, or Leſs than Juſt, &c.

Yet methinks I hear the young Learner ſay, it is impoſſible to follow the Directions and Examples, as they are here laid down; but ſtill here is not the Reaſon why they are ſo, and ſo, performed; and why there ſhould be a Remainder left after the Root is found, *viz.* when the given Reſolvend hath a true Root of its Kind.

It is true, the Reaſons of theſe are not here laid down; neither indeed can they be rendered ſo plain and intelligible by Words, as by an Algebraic Proceſs, from whence the *Theorems* or *Rules* here given, had their firſt Invention; as ſhall be ſhewed in the next Part, when I come to treat of reſolving compounded or adfected Æquations; however, take this ſhort and general Account of this Method.

This, and all other of the new Methods of Converging Series (as they are called) are very different from the former (and ſtill common) Methods of extracting Roots, which require the firſt ſingle Side or Root of the firſt Period (in any Reſolvend) to be taken exactly true, and then by involving, and other tedious Ways of ordering it, there is formed a Diviſor; which helps to grope out by Trials a ſecond Figure in the Root. And ſo proceed on from Point to Point; ſtill repeating the whole Work for every ſingle Figure that comes into the Root. And if by Chance there be a Miſtake or Error committed in any one Figure (as it is poſſible there may) it ſpoils the whole Proceſs, which muſt then be wholly begun anew, or at leaſt from that Part of it where the Error firſt entered.

But the Nature and Deſign of the Method which I have here laid down is quite otherwiſe; it being ſo contrived, as to gradually leſſen the Difference betwixt any propoſed Power, and the like Power of another Number aſſumed, *viz.* it leſſens that Difference until it is either quite vanquiſhed, or becomes ſo infinitely ſmall as to be inſignificant.

Therefore when any Number is propoſed to have its Root extracted, it is here required to take the next neareſt Root of the firſt Period in the Reſolvend; that ſo the Difference betwixt the given

given Resolvend, and the homogeneal Power (*viz.* the like Power) of the Root thus taken, may be less either in Excess, or Defect. Which Difference being reduced, or depressed lower, becomes so prepared, that by plain Division (comparatively) there will arise such Quotient Figures as will both correct and increase the first Root to three Places of Figures at least, sometimes to four, or five Places of Figures; according as the said first Difference happens to be more or less (of which you may have observed Instances): But yet there will be a Remainder left, and perhaps an Excess or Defect in the Root so increased, *viz.* in the last Figure of it.

Now to rectify the said Excess or Defect in the Root, and to discover whether the given Resolvend be a true figurative Number, or not: That is, whether it have a true Root of its Kind; it will be necessary to make a second Operation; by taking the Root so increased, and proceeding with it and the given Resolvend, in all Respects as in the first Work (like to the third *Example* of extracting the Cube Root); I say, if the given Resolvend have a true Root, it will appear at this second Operation, and all the aforesaid Differences, &c. will be vanquished; provided the Root required is not to have more than three, or four, Places of Figures in it.

But if the Root be to have more than three Figures in it, or that the given Resolvend prove to be a Surd Number. Then there will be a Difference as before; which will afford Quotient Figures to rectify and increase the Root last taken, to three times as many Places of Figures, as it had at the Beginning of that second Operation. As you may see in the aforesaid *Example* 3. of the Cube Root; wherein that Root is increased to twelve Places of Figures at two Operations; which if it were to be extracted the Old (and still common) Way, it would require at least forty Times the Number of Figures I have here used.

Again, if there chance to be a Mistake committed in any Operation performed by the Method here laid down, that Mistake will not destroy the precedent Work, but will be rectified in the next Operation, although it were not discovered before. And thus you may proceed on to a third Operation, which will afford 27 Places of Figures in the Root, &c. with very little Trouble, if compared with former Methods.

The brief Account which I have here given (*by Way of explaining the Nature of this Method of extracting Roots*) being well considered and compared with the several Operations of the foregoing *Examples*, must needs help the Learner to form such an Idea of it, that he cannot (I presume) but understand how to proceed

proceed in extracting the Root out of any single Power, how high soever it be, without the Help of an *Algebraic Theorem.* Not but when that comes to be once understood, the Work will be much readier and easier performed; as will appear in the next Part.

I did intend to have here inserted the whole Business of *Interest* and *Annuities*; but finding that it would require too large a Discourse, to shew the Grounds and Reasons for the several *Theorems* useful therein, I have therefore reserved that Work for the Close of the next Part. Neither indeed can the raising of those *Theorems* be so well delivered in Words, as by an *Algebraic* Way of arguing; which renders them not only much shorter, but also plainer and easier to be understood.

I have also omitted that *Rule* in *Arithmetick* usually called the *Rule of Position*, or *Rule of False:* Because all such Questions as can be answered by that guessing *Rule*, are much better done by any one who hath but a very small Smattering of *Algebra.* I shall therefore conclude this Part of *Numerical Arithmetick*, and proceed to that of *Algebraic Arithmetick*, wherein I would advise the young Learner not to be too hasty in passing from one *Rule* to another, and then he will find it very easy to be attainted.

AN

INTRODUCTION

TO THE

MATHEMATICKS.

PART II.

PROËM.

HAVING formerly wrote a ſmall Tract of ALGEBRA, *perhaps it may ſeem (to ſome) very improper to write again upon the ſame Subject; but only (as the uſual Cuſtom is) to have referred my Reader to that Tract. However, becauſe the following Parts of this Treatiſe are managed by an Algebraic Method of arguing; which may fall into the Hands of thoſe who have not ſeen that Tract, or any other of that Kind; I thought it convenient to accommodate the young Geometer with the firſt Elements, or principal Rules, by which all Operations in this Art are performed; that ſo he may not be at a Loſs as he proceeds farther on: Beſides, what I formerly wrote was only a Compendium of that which is here fully handled at large.*

The principal *Rules* are ADDITION, SUBSTRACTION, MULTIPLICATION, DIVISION, INVOLUTION, and EVOLUTION, as in common ARITHMETICK but differently performed; and therefore ſome call it ALGEBRAIC ARITHMETICK. Others call it ARITHMETICK IN SPECIE, becauſe all the Quantities concerned in any Queſtion, remain in their ſubſtituted Letters (howſoever managed by *Addition*, *Subſtraction*, or *Multiplication*, &c.) without being deſtroyed or changed into others, as Figures in common *Arithmetick* are.

Mr. *Harriot* called it LOGISTICA SPECIOSA, or *Specious Computation.*

CHAP.

CHAP. I.

Concerning the Method *of* Noting *down* Quantities; *and* Tracing *their* Steps, &c.

Sect. 1. *Of* Notation.

THE Method of noting down Letters for Quantities, is various, according to every one's Fancy; but I shall here follow the same as in my former Tract, and represent the Quantity sought (be it Line or Number, &c.) by the small (*a*,) and if more Quantities than one are sought, by the other small Vowels, *e. u.* or *y*.

The given Quantities are represented by the small Consonants, *b. c. d. f. g.* &c.

And for Distinction sake, mark the Points or Ends of Lines in all Schemes, with the capital or great Letters, *viz. A. B. C. D.* &c.

When any Quantity (either given or sought) is taken more than once, you must prefix its Number to it; as 3 *a* stands for *a* taken three times, or three times *a*, and 7 *b* stands for seven times *b*, &c.

All Numbers thus prefixed to any Quantity, are called Coefficients or Fellow-Factors; because they multiply the Quantity; and if any Quantity be without a Coefficient, it is always supposed or understood to have an Unit prefixed to it; as *a* is 1 *a*, or *b* is 1 *b*, &c.

The Signs by which Quantities are chiefly managed are the same, and have the same Signification, with those in the first Part, *page* 5. which I here presume the Reader to be very well acquainted with. To them must be here added these three more,

Viz. $\left\{\begin{matrix} ⓖ \\ ω \\ \surd \end{matrix}\right\}$ the Sign of $\left\{\begin{matrix}\text{Involution.} \\ \text{Evolution, or extracting Roots.} \\ \text{Irrationality, or Sign of a Surd Root.}\end{matrix}\right.$

All Quantities that are expressed by Numbers only (as in *Vulgar Arithmetick*) are called *Absolute Numbers*.

Those Quantities that are represented by single Letters, as, *a. b. c. d.* &c. or by several Letters that are immediately joined together, as *ab. cd.* or 7 *bd.* &c. are called Simple or Single whole Quantities.

But when different Quantities represented by different or unlike Letters, are connected together by the Signs (+ or —); as *a*+*b*, *a*—*b*, or *ab*—*dc*, &c. they are called Compound whole Quantities.

And

And when Quantities are expreſſed or ſet down like Vulgar Fractions, Thus $\frac{a}{b}$, or $\frac{a+b}{d}$, or $\frac{ab+dc}{b-c}$, &c. they are called Fractional or broken Quantities.

The Sign wherewith Quantities are connected, always belongs to that Quantity which immediately follows it; and therefore all the Quantities concerned in any Queſtion, may ſtand in any Order at Pleaſure, *viz.* the moſt convenient for the next Operation. As $a+b-d$ may ſtand thus $b-d+a$, or thus $a-d+b$, or $-d+a+b$ &c. theſe being ſtill the ſame, though differently placed.

That Quantity which hath no Sign before it (as generally the leading Quantity hath not) is always underſtood to have the Sign $+$ before it. As a is $+a$, or $b-d$ is $+b-d$, &c. for the Sign $+$ is the Affirmative Sign, and therefore all leading or poſitive Quantities are underſtood to have it, as well as thoſe that are to be added.

But the Sign — being the negative Sign, or Sign of Defect, there is a Neceſſity of prefixing it before that Quantity to which it belongs, wherever the Quantity ſtands.

Sect. 2. *Of* TRACING *the* STEPS *uſed in bringing* QUANTITIES *to an* EQUATION.

THE Method of tracing the Steps, uſed in bringing the Quantities concerned in any Queſtion to an Equation, is beſt performed by regiſtering the ſeveral Operations with Figures and Signs placed in the Margin of the Work, according as the ſeveral Operations require; being very uſeful in long and tedious Operations.

For Inſtance: If it be required to ſet down, and regiſter the Sum of the two Quantities, a and b, the Work will ſtand,

Thus	1	a
	2	b
$1+2$	3	$a+b$

Firſt ſet down the propoſed Quantities, a and b, over-againſt the Figures 1, 2, in the ſmall Column, (which are here called Steps) and againſt 3 (the third Step) ſet down their Sum, *viz.* $a+b$. Then againſt that third Step, ſet down $1+2$ in the Margin; which denotes that the Quantities againſt the firſt and ſecond Steps are added together, and that thoſe in the third Step are their Sum.

To illuſtrate this in Numbers, ſuppoſe $a=9$ and $b=6$. Then it will be,

Thus	1	$a=9$
	2	$b=6$
$1+2$	3	$a\ b=9+6=15$ being the Sum of 9 and 6.

Again, If it were required to set down the Difference of the same two Quantities; then it will be,

Thus	1	$a=9$
	2	$b=6$
$1-2$	3	$a-b=9-6=3$ the Difference between 9 and 6.

Or if it were required to set down their Product. Then it will be,

Thus	1	$a=9$
	2	$b=6$
1×2	3	$a\times b$ or $ab=9\times 6=54$ the Product of 9 into 6. &c.

Note, *Letters set or joined immediately together (like a Word) signify the Rectangle or Product of those Quantities they represent; as in the last* Example, *wherein* $ab=54$ *is the Product of* $a=9$ *and* $b=6$, &c.

Axioms.

1. If equal Quantities be added to equal Quantities, the Sum of these Quantities will be equal.
2. If equal Quantities be taken from equal Quantities, the Quantities remaining will be equal.
3. If equal Quantities be multiplied with equal Quantities, their Products will be equal.
4. If equal Quantities be divided by unequal Quantities, their Quotients will be equal.
5. Those Quantities, that are equal to one and the same Thing, are equal to one another.

Note, *I advise the Learner to get these five Axioms perfectly by Heart.*

These Things being premised, and a perfect Knowledge of the Signs and their Significations being gained, the young *Algebraist* may proceed to the following *Rules*. But first I must make bold to advise him here (as I have formerly done) that he be very ready in one *Rule* before he undertakes the next.

That is, he should be expert in *Addition*, before he meddles with *Substraction*; and in *Substraction*, before he undertakes *Multiplication*, &c. because they have a Dependency one upon another.

CHAP. II.

Concerning the Six principal RULES, *of* ALGEBRAIC ARITHMETICK, *of whole* QUANTITIES.

Sect. I. ADDITION of whole QUANTITIES.

ADDITION of whole Quantities admits of three Cases.

Case 1. If the Quantities are like, and have like Signs, add the Co-efficients or prefixed Numbers together, and to their Sum adjoin the Quantities with the same Sign.

		Exam. 1.	*Exam.* 2.	*Exam.* 3.	*Exam.* 4.
	1	a	$-a$	$5b$	$-7bc$
	2	a	$-a$	$3b$	$-8bc$
$1+2$	3	$2a$	$-2a$	$8b$	$-15bc$

Thus		*Exam.* 5.	*Exam.* 6.	*Exam.* 7.
	1	$3a+5b$	$3a-5b$	$6ab+12$
	2	$2a+7b$	$2a-7b$	$3ab+24$
$1+2$	3	$5a+12b$	$5a-12b$	$9ab+36$

The Reason of these Additions is evident from the Work of common Arithmetick. *For suppose* a, *to represent one Crown, to which if I add one Crown, the Sum will be two Crowns, or* 2a *as in Ex.* 1.

Or if we suppose —a, *to represent the Want or Debt of one Crown, to which if another Want or Debt of one Crown be added, the Sum must needs be the Want or Debt of two Crowns, or* —2a; *as in Example* 2. *And so for all the rest.*

Case 2. If the Quantities are like, and have unlike Signs; substract the Co-efficients from each other, and to their Difference join the Quantities with the Sign of the greater.

		Exam. 8.	*Exam.* 9.	*Exam.* 10.	*Exam.* 11.
	1	$+5a$	$-5a$	$7bc$	$-9abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
$1+2$	3	$+2a$	$-2a$	bc	$-2abd$

		Exam. 12.	*Exam.* 13.
	1	$7a-5b$	$-8ab-7bc+15$
	2	$-5a+7b$	$+12ab+7bc-24$
$1+2$	3	$2a+2b$	$4ab-9$

The Reason of the Operations in this Case may be easily understood by any one that duly considers the comparing of Stock and Debts together, or the balancing of Accounts betwixt Debtor and Creditor. That is, the affirmative Quantities represent the Stock or Creditor: The negative Quantities represent the Debts; and their Sum represents the Balance, &c.

Case 3. When the Quantities are unlike, set them all down, without altering their Signs; and thence will arise compound Quantities, which can be no otherwise added but by their Signs.

Thus				
	1	a	a	$5b+7dc$
	2	b	$-b$	$4a-20$
$1+2$	3	$a+b$	$a-b$	$5b+7dc+4a-20$

Here follow a few Examples wherein all the 3 *Cases* are promiscuously concerned.

	1	$aa+2ab+bb$	$8ab+bc-37$
	2	$-4ab$	$-7ab-bc+42-6d$
$1+2$	3	$aa-2ab+bb$	$ab+5-6d$

	1	$aa-2ab+bb$	$9bc+7ab-45$
	2	$+4ab$	$4d-6bc-7ab+da$
$1+2$	3	$aa+2ab+bb$	$3bc+4d-45+da$

	1	$5a$	$a+b-ab$
	2	$-7a$	$7c-d$
	3	$+3a$	$4e+f$
$1+2+3$	4	a	$a+b-ab+7c-d+4e+f$

	1	$3aa+4abc-bb+30$
	2	$2bb-3aa-2abc-25$
	3	$dd+2aa-3abc-3$
$1+2+3$	4	$+dd-2aa+bb-abc+2$

Sect. 2. SUBSTRACTION of whole QUANTITIES.

SUBSTRACTION of whole Quantities is performed by one general Rule.

RULE.

Change all the Signs of the Subtrahend, (viz. *of those Quantities which are to be substracted) or suppose them in your Mind to be changed; then add all the Quantities together, as before in* Addition, *and their Sum will be the true Remainder or Difference required.*

This

This general Rule is deduced from theſe evident Truths.

To ſubſtract an affirmative Quantity, from an Affirmative, is the ſame as to add a negative Quantity to an Affirmative: that is $+2a$ taken from $+3a$, is the ſame with $-2a$ added to $+3a$. Conſequently, to ſubſtract a negative Quantity from an Affirmative, will be the ſame as to add an affirmative Quantity to an Affirmative: that is $-2a$ taken from $+3a$ will be the ſame with $+2a$ added to $+3a$.

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
	1	$2a$	$-2a$	$8b$	$-15bc$
	2	a	$-a$	$3b$	$-8bc$
1—2	3	a	$-a$	$5b$	$-7bc$

		Exam. 5.	Exam. 6.	Exam. 7.
	1	$5a+12b$	$5a-12b$	$9ab+36$
	2	$2a+7b$	$2a-7b$	$3ab+24$
1—2	3	$3a+5b$	$3a-5b$	$6ab+12$

		Exam. 8.	Exam. 9.	Exam. 10.	Exam. 11.
	1	$+2a$	$-2a$	bc	$-2abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
1—2	3	$+5a$	$-5a$	$+7bc$	$-9abd$

		Exam. 12.	Exam. 13.
	1	$2a+2b$	$4ab-9$
	2	$-5a+7b$	$-8ab-7bc+15$
1—2	3	$7a-5b$	$12ab+7bc-24$

If theſe 13 *Examples* be compared with thoſe in *Addtion*, the Work will appear very evident, theſe being only the Converſe or Proof of thoſe; according to the Nature of *Addition* and *Subſtraction* in common *Arithmetick*.

More *Examples* in *Subſtraction*.

1	$a+b$	$5bc+3da$	$8a+5bd+25$
2	$a-b$	$5bc-4da$	$7a-3bd-12$
3	$+2b$	$+7da$	$a+8bd+37$

1—2

	1	$c+13$	a	0
	2	$3a-b-2c$	b	$2a-4b$
$1-2$	3	$3c+13-3a+b$	$a-b$	$-2a+4b$

	1	$a+b-54$	76
	2	$d-3b-bc-75$	$a-b-5d+7c$
$1-2$	3	$a+4b+bc+21-d$	$76-a+b+5d-7c$

That $a-b$ taken from $a+b$ leaves $+2b$ for the Remainder, as in the first of these *Examples*, may be thus proved:

Let	1	$a+b=z$	
And	2	$a-b=x$	
$2+b$	3	$a=x+b$	*per Axiom* 1.
$1-3$	4	$b=z-x-b$	*per Axiom* 2.
$4+b$	5	$2b=z-x$ which was to be proved.	

The Truth of all Operations in *Substraction*, where any Doubt arises, may be proved, by adding the Subtrahend to the Remainder, as in common *Arithmetick*.

EXAMPLE.

From	1	$+5a$	0	$-9bc$	
Take	2	$-2a$	$+3b$	$-6da$	Subtrahend.
$1-2$	3	$+7a$	$-3b$	$+6da-9bc$	Remainder.
$2+3$	4	$+5a$	0	$9bc$	Proof.

Sect. 3. MULTIPLICATION of whole QUANTITIES.

MULTIPLICATION of whole Quantities admits of three Cases.

Case 1. When the Quantities have like Signs, and no Coefficients, set or join them together, and prefix the Sign + before them; and that will be their Product.

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
Thus	1	a	$-a$	$a+b$	$-a-b$
	2	b	$-b$	d	$-d$
$1+2$	3	ab	$+ab$	$ad+bd$	$+ad+bd$

Case 2. If there be Coefficients; multiply them, and to their Product adjoin the Quantities set together as before.

Thus

		Exam. 5.	Exam. 6.	Exam. 7.	Exam. 8.
Thus	1	$5a$	$-6d$	$3a+2b$	$a+b$
	2	$3b$	$-7b$	6	$5b$
1×2	3	$15ab$	$+42db$	$18a+12b$	$5ab+5bb$

Case 3. When the Quantities have unlike Signs, join them and the Product of their Coefficients together (as before) but prefix the Sign —, before them;

		Exam. 9.	Exam. 10.	Exam. 11.	Exam. 12.
Thus	1	$+a$	$-6d$	$4a-7b$	$4a-7b$
	2	$-b$	$+7b$	$3f$	$-3f$
1×2	3	$-ab$	$-42db$	$12af-21bf$	$-12af+21bf$

That is, + into +, or — into —, gives + } in the Product.
But + into —, or — into +, gives — }

That + into + will produce + in the Product is evident from *Multiplication* in common *Arithmetick*: viz. $+5$ into $+7$ will give $+35$, &c. But that + into —, or — into + should produce the Sign —, as in the four last Examples; and that — into — should produce the Sign +, as in the second, fourth, and sixth Examples, may perhaps seem somewhat hard to be conceived; and requires a Demonstration.

First to prove that $-7b$ into $+3f=-21bf$. As in *Ex.* 11.

Suppose	1	$4a-7b=0$	
Then will	2	$4a=7b$	*per Axiom* 1.
But	3	$+3f=+3f$	
2×3	4	$12af=21bf$	*per Axiom* 3.
$4-21bf$	5	$12af-21bf=0$	*per Axiom* 2.

Consequently + into —, or — into + produces —, which was the Thing to be proved.

Secondly to prove that $-7b$ into $-3f$ gives $+21bf$ as in *Example* 12.

Let	1	$4a-7b=0$	} as before,
Then	2	$4a=7b$	}
But	3	$-3f=-3f$	
the 2×3 is	4	$-12af=-21bf$	by what is proved above.
4 21 bf	5	$-12af+21bf=0$.	*per Axiom* 1.

Consequently — into — gives + which was to be proved.

Or

Or these may be otherwise proved by Numbers.

Thus, suppose $\left\{\begin{matrix} a=20 \\ b=14 \end{matrix}\right\}$ and $\left\{\begin{matrix} c=12 \\ d=\ 8 \end{matrix}\right\}$ $\Big\{$ or any other Numbers.

Then $\underline{a-b=6}$ $\underline{c-d=4}$ *per Axiom* 2.

Consequently, $\overline{a-b}\times\overline{c-d}=6\times4=24$, *per Axiom* 3. but $\overline{a-b}\times\overline{c-d}$, according to the precedent Rules, will be, $ac-cb+bd-da$, which if true must be equal to 24.

$$\text{Proof}\left\{\begin{matrix} ac=20\times12=240 & cb=12\times14=168 \\ bd=14\times\ 8=112 & da=\ 8\times20=160 \end{matrix}\right.$$

Hence $ac+bd=352$ *per Axiom* 1.

And $cb+da=328$ which being substracted,

Leaves $ac+bd-cb-da=352-328=24$, which plainly shews,

That + into — produces — }
And — into — produces + } in the Product.

Q. E. D.

Note, If the Multiplier consists of several Terms, then every one of those Terms must be multiplied into all the Terms of the Multiplicand; and the Sum of those particular Products, will be the Product required, as in common *Arithmetick*.

EXAMPLES.

	1	$a+b-d$	$7b+5d$
	2	$a-b$	$3a-5f$
$1\times a$	3	$aa+ba-da$	$21ba+15da$
$1\times b$	4	$-ba-bb+db$	$-35bf-25df$
$3+4$	5	$aa-da-bb+db$	$21ba+15da-35bf-25df$

	1	$aa-ba$	$2c-3d$
	2	$a+b$	$3a-4b$
1×2	3	$aaa-abb$	$6ca-9da-8bc+12ab$

	1	$aa+2a+4$	$aa-ba+bb$
	2	$a-2$	$a+b$
		$aaa+2aa+4a$	$aaa-baa+bba$
		$-2aa-4a-8$	$+baa-bba+bbb$
1×2	3	$aaa-8$	$aaa+bbb$

Sect.

Sect. 4. Division *of whole* Quantities.

Division of Species, is the converse or direct contrary to that of *Multiplication*, and consequently is performed by converse Operations, (as in common *Arithmetic*) and admits of four *Cases*.

Case 1. When the Quantities in the Dividend, have like Signs to those in the Divisor, and no Co-efficients in either; cast off or expunge all the Quantities in the Dividend, that are like those in the Divisor; and set down the other Quantities with the Sign + for the Quotient required.

Thus					
	1	ab	$-ab$	$ad+bd$	$-ad-bd$
	2	b	$-b$	d	$-d$
1÷2	3	a	$+a$	$a+b$	$a+b$

Case 2. When the Quantities in the Dividend have unlike Signs to those in the Divisor; then set down the Quotient Quantities found as before, with the Sign — before them.

Thus				
	1	$+ab$	$-ab-bd$	$abc+bcd+bcf$
	2	$-b$	$+b$	$-bc$
1÷2	3	$-a$	$-a-d$	$-a-d-f$

Case 3. If the Quantities in the Dividend and Divisor, have Co-efficients; divide the Number (as in common *Arithmetick*) and to their Quotient adjoin their Quotient Quantities.

Thus				
	1	$15ab$	$42db$	$12af-21bf$
	2	$3b$	$-7b$	$3f$
1÷2	3	$5a$	$-6d$	$4a-7b$

Note, When the Quantities and Co-efficients in the Divisor and Dividend are all the same, the Quotient will be an Unit, or 1.

Thus					
	1	ab	$9bc$	$7ab+5bc$	$8ab+4d$
	2	ab	$-9bc$	$7ab+5bc$	$-8ab-4d$
1÷2	3	1	-1	1	-1

Case 4. When the Quantities in the Divisor cannot be exactly found in the Dividend; then set them both down like a *Vulgar Fraction* as in common *Arithmetick*.

Thus					
	1	a	$6\,bc$	$5\,b+a\,a$	$8\,adc$
	2	b	$3\,d$	$5\,d+7\,b$	$4\,abc$
$1 \div 2$	3	$\frac{a}{b}$	$\frac{2\,bc}{d}$	$\frac{5\,b+a\,a}{5\,d+7\,b}$	$\frac{2\,d}{b}$

N. B. In Division one thing must be very carefully observed; *viz.* that like Signs give + and unlike Signs give — in the Quotient; which needs no other Proof than that already laid down in the last Section, if duly compared with what hath been said concerning *Multiplication* and *Division*, in Vulgar *Arithmetick*.

Examples of *Division* at large.

	1	$21\,ba + 15\,da - 35\,bf - 25\,df\ (+ 3a$
	2	$7\,b + 5\,d$
$2 \times 3\,a$	3	$21\,ba + 15\,da$
$1 - 2$	4	$0 \quad 0 \quad - 35\,bf - 25\,df\ (- 5f$
$2 \times - 5f$	5	$- 35\,bf - 25\,df$
$4 - 5$	6	$0 \quad 0$
$1 \div 2$	7	$3a - 5f$ the Quotient collected from the 3, and 5 Steps.

Or *Division* of Quantities may stand as Numbers in common *Arithmetick* do: thus

$$
\begin{array}{l}
3\,a - 6)\ 6\,aaaa - 96\ (2\,aaa + 4\,aa + 8\,a + 16 \\
\qquad\quad 6\,aaaa - 12\,aaa \\
\hline
\qquad\quad 0 + \quad 12\,aaa - 96 \\
\qquad\qquad + \quad 12\,aaa - 24\,aa \\
\hline
\qquad\qquad\qquad 0 \quad + 24\,aa - 96 \\
\qquad\qquad\qquad\quad + 24\,aa - 48\,a \\
\hline
\qquad\qquad\qquad\qquad 0 \quad + 48\,a - 96 \\
\qquad\qquad\qquad\qquad\quad + 48\,a - 96 \\
\hline
\qquad\qquad\qquad\qquad\quad 0 \quad 0
\end{array}
$$

That is, $\overline{6\,aaaa - 96} \div \overline{3a - 6}$ gives $2\,aaa + 4\,aa + 8\,a + 16$ for the Quotient, as may easily be proved by *Multiplication*, viz. $\overline{2\,aaa + 4\,aa + 8\,a + 16} \times \overline{3\,a - 6}$ will produce $6\,a^4 - 96$; and so for the rest.

Sect. 5. Involution *of whole* Quantities.

INvolution is the raising or producing of Powers, from any proposed Root, and is performed in all respects like *Multiplication*, save only this; *Multiplication* admits of any different Factors, but Involution still retains the same.

EXAMPLES.

EXAMPLES.

	1	a	$-a$	the Root, or single Power.
1 ©.2	2	aa	$+aa$	Square, or second Power.
1 ©.3	3	aaa	$-aaa$	Cube, or third Power.
1 ©.4	4	$aaaa$	$+aaaa$	Biquadrat, or fourth Power.
1 ©.5	5	$aaaaa$	$-aaaaa$	Surfolid, or fifth Power, &c.

Note, The Figures placed in the Margin, after the Sign (©.) of Involution, shew to what Height the Root is involved; and are called Indices of the Power; and are usually placed over the involved Quantities in order to contract the Work, especially when the Powers are any thing high.

Thus $\begin{cases} a = a \\ a^2 = aa \\ a^3 = aaa \\ a^4 = aaaa \end{cases}$ And $\begin{cases} a^5 = aaaaa \\ a^6 = aaaaaa \\ a^5\, b^5 = aaaaabbbbb \\ a^3\, b^3\, d^3 = aaabbbddd \end{cases}$

If the Quantities have Co-efficients, the Co-efficients must be involved along with the Quantities, as in these,

Thus	1	$2\,a$	$-3\,a$	$5\,bc$
1 ©.²	2	$4\,aa$	$+9\,aa$	$25\,bbcc$
1 ©.³	3	$8\,aaa$	$-27\,aaa$	$125\,bbbccc$
1 ©.⁴	4	$16\,aaaa$	$+81\,aaaa$	$625\,b^4\,c^4$
1 ©.⁵	5	$32\,aaaaa$	$-243\,a^5$	$3125\,b^5\,c^5$, &c.

Involution of Compound Quantities is performed in the same manner, due regard being had to their Signs and Co-efficients, if there be any. As for instance, suppose $a+b$ were given to be involved to the fifth Power.

Thus	1	$a+b$ called a Binomial Root.
		$a+b$
1 × a	2	$aa+ab$
1 × b	3	$\quad +ab+bb$
1 ©.²	4	$aa+2\,ab+bb$, the Square of $a+b$
		$a+b$
4 × a	5	$aaa+2\,aab+abb$
4 × b	6	$\quad + aab+2abb+bbb$
1 ©.³	7	$aaa+3aab+3abb+bbb$, the Cube of $a+b$

	7	$aaa+3aab+3abb+bbb$
		$a+b$
$7\times a$	8	$a^4+3a^3b+3aabb+abbb$
$7\times b$	9	$+a^3b+3aabb+3abbb+b^4$
$1 ⊚ 4$	10	$a^4+4a^3b+6aabb+4abbb+b^4$
		$a+b$
$10\times a$	11	$a^5+4a^4b+6a^3bb+4aab^3+ab^4$
$1\times b$	12	$a^4b+4a^3bb+6aab^3+4ab^4+b^5$
$1 ⊚ 5$	13	$a^5+5a^4b+10a^3bb+10aab^3+5ab^4+b^5$
		&c.

Again, Let $a-b$, called a Residual Root, be given.

Then	1	$a-b$
		$a-b$
$1\times a$	2	$aa-ab$
$1\times -b$	3	$-ab+bb$
$1 ⊚ 2$	4	$aa-2ab+bb$, the Square of $a-b$
		$a-b$
$4\times a$	5	$aaa-2aab+abb$
$4\times -b$	6	$-aab+2abb-bbb$
$1 ⊚ 3$	7	$aaa-3aab+3abb-bbb$, the Cube of $a-b$
		$a-b$
$7\times a$	8	$aaaa-3aaab+3aabb-abbb$
$7\times -b$	9	$-aaab+3aabb-3abbb+bbbb$
$1 ⊚ 4$	10	$aaaa-4aaab+6aabb-4abbb+bbbb$
		$a-b$
$10\times a$	11	$a^5-4a^4b+6a^3bb-4aab^3+ab^4$
$10\times -b$	12	$-a^4b+4a^3bb-6aab^3+4ab^4-b^5$
$1 ⊚ 5$	13	$a^5-5a^4b+10a^3bb-10aab^3+6ab^4-b^5$
		&c.

By comparing these two Examples together, you may make the following Observations.

1. That the Powers raised from a Residual Root (*viz.* the Difference of two Quantities) are the same with their like Powers raised from a Binomial Root (or the Sum of two Quantities) save only in their Signs; *viz.* the Binomial Powers have the Sign + to every Term, but the Residual Powers have the Signs + and — interchangeably to every other Term.

2. The Indices of the Powers of the leading Quantity (a) continually decrease in Arithmetical Progression; *viz.* in the Square

Square it is *a a*, *a*: In the Cube *aaa*, *aa*, *a*: In the Biquadrat *aaaa*, *aaa*, *aa*, *a*, &c.

3. The Indices of the other Quantity *b* do continually increase in Arithmetical Progression; *viz.* In the Square it is *b*, *bb*: In the Cube *b*, *bb*, *bbb*: In the Biquadrat *b*, *bb*, *bbb*, *bbbb*, &c.

4. The first and last Terms, are always pure Powers of the single Quantities, and are both of the same Height.

5. The Sum of the Indices of any two Letters joined together in the intermediate Terms, are always equal to the Index of the highest Power, *viz.* of the first or last Term.

These Observations being duly considered, it will be easy to conceive how the Terms of any proposed Power raised from a Binomial or Residual Root must stand, without their Unciæ or Numeral Figures.

For Instance, suppose it were required to raise the Binomial Root $a+b$ to the seventh Power; then the Terms of that Power will stand without the Unciæ in this Order.

Viz. $a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$.

And because the Unciæ (not only of any single Letter, but also) of every single Power, how high soever it be, is an Unit or 1 (which neither multiplies nor divides) and all the Powers of any Binomial or Residual Root are naturally raised by multiplying of the precedent Power into its original Root, which is done by only joining each Letter in the Root to the precedent Power, with it's Unciæ, and then removing the said Power, when it is so joined to the second Letter, one Place forward (either to the Left or right Hand) it must needs follow,

That the Unciæ of the second Terms (in any such Power) will always be the Sum of so many Units added together more one, as there have been Multiplications of the first Root: which will always be determined by the Index of the first Term in the Power.

And because the Unciæ of all the intermediate Terms, are only removed along with their Letters, it also follows, that if they are added together, their respective Sums will produce the true Unciæ of the intermediate Terms in the new raised Power. As doth plainly appear from the following Numbers so removed without their Letters; which both shews and demonstrates an easy Way of producing the Unciæ of any ordinary Power (*viz.* of not one very high) raised from either a Binomial or Residual Root.

Thus

Thus

$$
\begin{array}{l}
\text{Add}\left\{\begin{array}{llllllll}1\,. & 1\,. & & & & & & \\ & 1\,. & 1 & & & & & \end{array}\right. \quad \text{The two Unciæ of the Root.} \\
\hline
\text{Add}\left\{\begin{array}{llllllll}1\,. & 2\,. & 1 & & & & & \\ & 1\,. & 2\,. & 1 & & & & \end{array}\right. \quad \text{The Unciæ of the Square.} \\
\hline
\text{Add}\left\{\begin{array}{llllllll}1\,. & 3\,. & 3\,. & 1 & & & & \\ & 1\,. & 3\,. & 3\,. & 1 & & & \end{array}\right. \quad \text{The Unciæ of the Cube.} \\
\hline
\text{Add}\left\{\begin{array}{llllllll}1\,. & 4\,. & 6\,. & 4\,. & 1 & & & \\ & 1\,. & 4\,. & 6\,. & 4\,. & 1 & & \end{array}\right. \quad \text{The Unciæ of the 4th Power.} \\
\hline
\text{Add}\left\{\begin{array}{llllllll}1\,. & 5\,. & 10\,. & 10\,. & 5\,. & 1 & & \\ & 1\,. & 5\,. & 10\,. & 10\,. & 5\,. & 1 & \end{array}\right. \quad \text{Unciæ of the fifth Power.} \\
\hline
\text{Add}\left\{\begin{array}{llllllll}1\,. & 6\,. & 15\,. & 20\,. & 15\,. & 6\,. & 1 & \\ & 1\,. & 6\,. & 15\,. & 20\,. & 15\,. & 6\,. & 1\end{array}\right. \quad \text{Unciæ of the 6th Pow.} \\
\hline
\quad 1\,.\ 7\,.\ 21\,.\ 35\,.\ 35\,.\ 21\,.\ 7\,.\ 1 \quad \text{Unciæ of 7th Pow.}
\end{array}
$$

And so on in this Manner *ad infinitum.*

Now if these Numbers are prefixed to the aforesaid Letters, all the Terms will be compleated with their respective Unciæ, and will stand thus;

$$a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b.$$

But that the Business of finding these Unciæ, may be rendered yet more easy for Practice, it will be convenient to consider what Series or Progression, the Unciæ of each Term do make from the aforesaid Additions,

Unciæ of the first Term.	Unciæ of the second Term.	Unciæ of the third Term.	Unciæ of the 4th Term.	Unciæ of the fifth Term.	Unciæ of the sixth Term.	Unciæ of the 7th Term.	Unciæ of the 8th Term, &c.	
1	1	..	..	..	..	..	..	Unciæ of the single Quantities.
1	2	1	..	..	..	..	..	Unciæ of the Square.
1	3	3	1	..	..	..	..	Unciæ of the Cube.
1	4	6	4	1	..	..	..	Unciæ of the 4th Power.
1	5	10	10	5	1	..	..	Unciæ of the 5th Power.
1	6	15	20	15	6	1	..	Unciæ of the 6th Power.
1	7	21	35	35	21	7	1	Unciæ of the 7th Power, &c.

The Unciæ of the first Term are only a Series of Units, whose Sum is every where the Unciæ of the second Term. The Unciæ of the second Term, are a Series of Numbers in arithmetical Progression, whose Sum is every where the Unciæ of the next superior Power in the third Term, and may be found by Proposition

tion 1. Chap. 6 Part 1. For Inſtance, in the ſeventh Power it will be $\frac{\overline{6+1}\times 6}{2}=21=$ the Unciæ of the third Term.

The reſt of the Unciæ are a compounded Series, whoſe reſpective Sums may be obtained from the Unciæ of their precedent Terms.

Thus $\frac{21\times 5}{3}=35$. Then $\frac{35\times 4}{4}=35$. Again $\frac{35\times 3}{5}=21$. And $\frac{21\times 2}{6}=7$, &c.

From hence may be deduced this general Rule.

RULE.

If the Index of the firſt Letter of any Term be multiplied into its own Unciæ, and that Product be divided by the Number of Terms to that Place; the Quotient will be the Unciæ of the next ſucceeding Term forward.

That is, by the Help of thoſe Indices that belong to the ſeveral Powers of the firſt or leading Letter only (as *a*) the true Unciæ of every Term may be eaſily underſtood.

EXAMPLES.

Let it be required to compleat all the Terms of the aforeſaid ſeveral Powers, *viz.* $a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$, with their proper Unciæ.

1. The Index of a^7 the firſt Term will be the Unciæ of the ſecond Term. Thus a^7+7a^6b.

2. Then half the ſecond Term's Index into its Unciæ, *viz.* $\frac{7+6}{2}=21$ will be the third Term's Unciæ. Thus $a^7+7a^6b+21a^5b^2$ will be the three firſt Terms.

3. Again $\frac{21\times 5}{3}=35$ is the Unciæ of the fourth Term, whence, $a^7+7a^6+b+21a^5b^2+35a^4b^3$ will be the four firſt Terms.

4. And $\frac{35\times 4}{4}=35$ will be the Unciæ of the fifth Term, whence $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4$ will be the five firſt Terms.

And ſo proceed till all the Terms are compleated with their reſpective Unciæ, which will ſtand, thus $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

Now

Now here it may be further observed, that the Unciæ do only increase until the Indices of the two Letters become equal, or change Places; and then the rest of the Unciæ will return or decrease in the same Order. That is, wherever the Indices of the Letters are alike, there the Unciæ will be alike.

And therefore one needs to find the Unciæ (as before) but to half the Number of Terms in any Power.

If what hath been said, and the Work of the *Example* be well understood, I presume it will be found very easy to raise any Power from a Binomial or Residual Root, to what Height you please; without the Trouble of a continued Involution; and without the Help of such a Table of Powers as is proposed by Mr. *Oughtred* in his Key to the *Mathematicks*, Page 40, and since by others.

Now from these Considerations it was that I proposed this Method of raising Powers in my Compendium of *Algebra*, Page 57, as wholly New (*viz.* so much of it as was there useful) having then (I profess) neither seen the Way of doing it, nor so much as heard of its being done. But since the writing that Tract, I find in Dr. *Wallis's History of Algebra*, Page 319 and 331, that the learned Sir *Isaac Newton* had discovered it long before: which the Doctor sets down in this Manner.

Let m be the Exponent of the Power.

$$\text{Then}\left\{1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-1}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}\right.$$

Will be the Series of the Unciæ required; but he doth not tell us how they first came to be found out, nor have I ever met with the least Hint of it in any Author.

Sect. 6. Evolution *of whole* Quantities.

Evolution is the extracting of Roots from any given Power. That is, it is the Converse Work to that of Involution, and in single Quantities it is easy, if the given Power have such a Root as is required, which may be thus known.

If the given Power have no Numbers prefixed to it, and its Index can be divided by the Index of the Root required, the Quotient will be the Index of the Root sought. Thus, if the Cube Root of *aaaaaa*, viz. a^6 were required (the Index of the Cube is 3) then 3) 6 (2. That is, $a^{\frac{6}{3}} = a^2$ the Root required. And such Operations are usually set down

Thus

Thus	1	a^6	$a^6\,b^6$	$a^6\,b^6\,d^6$
1 ɯ^2	2	a^3	$a^3\,b^3$	$a^3\,b^3\,d^3$
1 ɯ^3	3	a	$a^2\,b^2$	$a^2\,b^2\,d^2$
3 ɯ^2	4	a	ab	abd

Note, The Figures placed in the Margin after the Sign (ɯ) of Evolution, denote the Index of the Root to be extracted.

If the given Powers have Co-efficients: *viz.* Numbers prefixed to them ;) then you must extract their respective Roots, as in Vulgar *Arithmetick.*

Thus	1	$81\,a^4$	$1296\,a^8\,b^8$	$20736\,a^4\,b^4\,c^4$
1 ɯ^2	2	$9\,a$	$36\,a^4\,b^4$	$144\,a^2\,b^2\,c^2$
1 ɯ^4	3	$3\,a$	$6\,a^2\,b^2$	$12\,abc$
or 2 ɯ^2	4	$3\,a$	$6\,a^2\,b^2$	$12\,abc$

But if the Root required cannot be truly extracted out of both the Co-efficients and Indices of the given Power; then it is a Surd, and must have the Sign of the Root required prefixed to it.

Thus	1	a^5	$67\,a^4$	$216\,bbbddd$
1 ɯ^2	2	$\sqrt{a^5}$	$\sqrt{67\,a^4}$	$\sqrt{216\,bbbddd}$
1 ɯ^3	3	$\sqrt[3]{a^5}$	$\sqrt[3]{67\,a^4}$	$6\,bd$

Evolution of Compound Quantities or Powers, is a little more troublesome than that of single Powers; and would require a great many Words to explain the Manner and Reason of forming the several Canons, that are commonly used in extracting the Roots of compound Quantities; especially if the Powers be very high, &c. I shall therefore for Brevity's Sake omit them, and instead thereof propose an easy Method of discovering the Roots of all compound Powers in general. And in order to that, it will be necessary to premise; that if either the Sum or Difference of several Quantities be involved to any Power, there will arise so many single Powers of the same Height, as there are different Quantities.

As for Instance, if $a+b+d$ be squared, that is, be involved to the second Power, it will be $aa+2ab+2ad+bb+2bd+dd$, here you have aa, bb, and dd. Again, if $a+b+d$ were cubed, *viz.* involved to the third Power, then you will have aaa, bbb, ddd, in it, &c.

Y Whence

Whence it follows that in extracting the Roots of all compound Quantities, there must be considered,

1. How many different Letters (or Quantities) there are in the given Power.

2. Whether the single Powers of each of those Letters be of an equal Height, and have in them such a single Root as is required: which if they have, extract it as before.

3. Connect those single Roots together with the Sign $+$, and involve them to the same Height with the given Power; that being done, compare the new raised Power with the given Power; and if they are alike in all their respective Terms, then you have the Root required; or if they differ only in their Signs, the Root may be easily corrected with the Sign $-$ as occasion requires.

Example 1. Let it be required to extract the Square Root of $cc+2cb-2cd+bb-2bd+dd$. In this Compound Square, there are three distinct Powers, *viz.* bb, cc, dd, whose single Roots are b, c, d, wherefore I suppose the Root sought to be $b+c+d$, or rather $b+c-d$, because in the given Power there is $-2cd$, and $-2bd$, therefore I conclude it is $-d$; then $b+c-d$, being squared, produces $bb+2bc-2bd+cc-2cd+dd$, which I find to be the same in all its Terms with the given Power, although they stand in a different Position; consequently $b+c-d$ is the true Root required.

Example 2. It is required to extract the Square Root of $a4-2aabb+b4$. Here are but two single Powers, *viz.* $a4$ and $b4$, whose Square Roots are aa and bb. And because in the given Power there is $-2aabb$, therefore I conclude it must either be $aa-bb$ or $bb-aa$. Both which, being involved, will produce $a4-2aabb+b4$; consequently the Root sought may either be $aa-bb$, or $bb-aa$ according to the Nature or Design of the Question from whence the given Power was produced.

Example 3. Let it be required to extract the Square Root of $36aaaa+108aa+81$. Here the two single Powers are $36aaaa$, and 81, whose Roots are $6aa$ and 9. And because the Signs are all $+$ therefore I suppose the Root to be $6aa+9$, the which being involved doth produce $36a4+108aa+81$; consequently $6aa+9$ is the true Root required.

Example 4. Suppose it were required to extract the Cube Root of $125aaa+300aae-450aa+250aee-720ae+64eee+540a-288ee+432e-216$. In this Example there are three distinct Powers, *viz.* $125aaa$, $64eee$, and -216. The Cube Root of $125aaa$ is $5a$; of $64eee$ is $4e$; And of -216 is -6. Wherefore I suppose the Root sought to be $5a+4e-6$, which

which being involved to the third Power, does produce the same with the given Power; consequently $5a+4e-6$ is the Cube Root required.

But if the new Power, raised from the supposed Root (being involved to its due Height) should not prove the same with the given Power, *viz.* if it hath either more or fewer Terms in it, &c. then you may conclude the given Power to be a Surd, which must have its proper Sign prefixed to it, and cannot be otherwise expressed, until it come to be involved in Numbers.

Example 5. Suppose it were required to extract the Cube Root of $27aaa + 54baa + 8bbb$. Here are two distinct and perfect Cubes, *viz.* $27aaa$, and $8bbb$, whose Cube Roots are $3a$ and $2b$. Wherefore one may suppose the Root sought to be $3a+2b$, which being involved to the third Power, is $27aaa + 54baa + 36bba + 8bb$. Now this new raised Power hath one Term (*viz.* $36bba$) more in it than the given Power hath; but this being a perfect Cube, one may therefore conclude the given Power is not so, *viz.* it is a Surd, and hath not such a Root as was required, but must be expressed, or set down,

Thus $\sqrt[3]{27aaa+54baa+8bbb}$.

If these Examples be well understood, the Learner will find it very easy by this Method of proceeding to discover the true Root of any given Power whatsoever.

CHAP. III.

Of Algebraick Fractions, *or* Broken Quantities.

Sect. 1. Notation *of Fractional Quantities.*

FRACTIONAL *Quantities* are expressed or set down like Vulgar Fractions in common *Arithmetick.*

$$\text{Thus}\left\{\begin{array}{l} \dfrac{a,}{b}\ \dfrac{2bc,}{d}\ \dfrac{5b-4a}{4d+7b} \end{array}\right. \begin{array}{l}\text{Numerators.}\\ \text{Denominators.}\end{array}$$

How they come to be so, see *Case* 4, in the last Chapter of *Division.* These Fractional Quantities are managed in all respects like Vulgar Fractions in Common *Arithmetick.*

Sect 2. *To* Alter *or* Change *different* Fractions *into one Denomination, retaining the same Value.*

RULE.

MULTIPLY all the Denominators into each other for a new Denominator, and each Numerator into all the Denominators but its own for new Numerators.

EXAMPLES.

Let it be required to bring $\frac{a}{b}$ and $\frac{d}{c}$ into one Denomination.

First $a\times c$, and $d\times b$, will be the Numerators, and $b\times c$ will be the common Denominator, *viz.* $\frac{ca}{bc}$ and $\frac{bd}{bc}$ are the two Fractions required: that is $\frac{ca}{bc}=\frac{a}{b}$, and $\frac{bd}{bc}=\frac{d}{c}$.

Again, let $\frac{b+c}{a+b}$ and $\frac{d-c}{b-d}$ be brought into one Denomination, and they will be $\frac{bb+bc-bd-dc}{ba+bb-da-bd}$, and $\frac{ad-ac+bd-bc}{ba+bb-da-bd}$, &c.

Sect. 3. *To* Bring *whole* Quantities *into* Fractions *of a given Denomination.*

RULE.

MULTIPLY the whole Quantities into the given Denominator for a Numerator, under which subscribe the given Denominator, and you will have the Fraction required.

EXAMPLES.

Let it be required to bring $a+b$ into a Fraction, whose Denominator is $d-a$. First $a+b\times d-a$ is $da+bd-aa-ba$: Then $\frac{da+bd-aa-ba}{d-a}$ is the Fraction required.

Again $b+\frac{a}{d}$ will be $\frac{db+a}{d}$. And $\frac{aa}{d}-a$ will be $\frac{aa-da}{d}$

Also $a+b+\frac{aa+bb}{a-b}$ will be $\frac{2aa}{a-b}$

When

When whole Quantities are to be ſet down Fraction-wiſe, ſubſcribe an Unit for the Denominator. Thus ab is $\frac{ab}{1}$ And $aa-bb$, is $\frac{aa-bb}{1}$, &c.

Sect. 4. *To* Abbreviate, *or* Reduce *Fractional Quantities into their loweſt Denomination.*

RULE.

Divide both the Numerator and Denominator by their greateſt common Diviſor, viz. *by ſuch Quantities as are found in both, and their Quantities will be the Fraction in its loweſt Term.*

Thus $\frac{aac}{dc}$ is $\frac{aa}{d}$. $\frac{abbb}{abc}$ is $\frac{bb}{c}$. And $a+\frac{bdc}{bc}=a+d$.

In ſuch ſingle Fractions as theſe, the common Diviſors (if there be any) are eaſily diſcovered by Inſpection only; but in compound Fractions it often proves very troubleſome, and muſt be done either by dividing the Numerator by the Denominator, until nothing remains, when that can be done: or elſe finding their common Meaſure, by dividing the Denominator by the Numerator, and the Numerator by the Remainder, and ſo on, as in Vulgar Fractions (Sect. 4. Page 51.)

EXAMPLES.

Suppoſe $\frac{aac-aad}{cd-dd}$ were to be reduced lower.

Then $cd-dd)\ aac-aad\left(\frac{aa}{d}\right.$ the Fraction required.

$$\begin{array}{l} aac-aad \\ \hline 0 \quad\ \ 0 \end{array}$$

In this Example it ſo happens that the Numerator is divided juſt off by the Denominator; but in the next it is otherwiſe, and requires a double Diviſion to find out the common Meaſure, *viz.*

Let it be required to reduce $\frac{aaa-abb}{aa+2ab+bb}$ to its loweſt Terms.

Firſt $aa+2ab+bb)\ aaa-abb\ (a$

$$\begin{array}{l} aaa+2aab+abb \\ \hline -2aab-2abb \end{array}$$ the Remainder.

Then $-2aab-2abb)\ aa+2ab+bb\ \left(-\frac{1}{2b}\ \frac{1}{2a}\right.$

$$\begin{array}{r} aa+ab \\ \hline ab+bb \\ ab+bb \\ \hline 0 \quad 0 \end{array}$$

Hence

Hence it appears that $-2aab-2abb$ is the common Measure, by which $aaa-abb$ being divided.

$$\begin{array}{rl} \textit{Viz. } -2aab-2abb) & aaa-abb \left(-\dfrac{a}{2b}+\dfrac{1}{2}\right. \\ & \underline{aaa+aab} \\ & -aab-abb \\ & \underline{-aab-abb} \\ & \quad 0 \quad\; 0 \end{array}$$

Then $-\frac{a}{2b}+\frac{1}{2}$ is the new Numerator; and ☞ $-\frac{1}{2b}$ $-\frac{1}{2a}$ is the new Denominator. But $-\frac{a}{2b}+\frac{1}{2}=\frac{-2a+2b}{4b}$ $=\frac{-a+b}{2b}$ the Numerator; and $-\frac{1}{2b}-\frac{1}{2a}=\frac{-2a-2b}{4ba}$ $=\frac{-a-b}{2ba}$ the Denominator. Let both be multiplied with $2ba$, and you will have $\frac{-aa+ab \text{ the Numerator.}}{-a-b \text{ the Denominator.}}$ Or changing the Signs of all the Quantities, it will be $\frac{aa-ab}{a+b}$ the new Fraction required. That is, $\frac{aa-ab}{a+b}=\frac{aaa-abb}{aa+2ab+bb}$

Again, let it be required to reduce $\frac{dd-bb}{ddd-bbb}$

The common Measure of this Fraction will be the easiest found (as appears from Trials) by dividing the Denominator by the Numerator, &c. Thus,

$$\begin{array}{llll} dd-bb) & ddd-bbb\,(d & & \\ & \underline{ddd-bbd} & & \\ & +bbd-bbb) & dd-bb\left(\dfrac{d}{bb}\right. & \\ & & \underline{dd-bd} & \\ & & +bd-bb) & bbd-b^3\,(b \\ & & & \underline{bbd-b^3} \\ & & & \;0 \quad\; 0 \end{array}$$

Hence it appears that $bd-bb$ is the common Measure that will divide both the Numerator and the Denominator.

Consequently

Consequently $bd-bb)\,dd-bb\,(\frac{d}{b}+1$, is the new Numerator.

$$\begin{array}{l} dd-db \\ \hline +db-bb \\ db-bb \\ \hline 0 \quad\ 0 \end{array}$$

And $bd-bb)\,ddd-bbb\,(\frac{dd}{b}+d+b$ the new Denominator.

$$\begin{array}{l} ddd-ddb \\ \hline +ddb-bbb \\ ddb-bbd \\ \hline +bbd-bbb \\ bbd-bbb \\ \hline 0 \quad\ 0 \end{array}$$

Let both be multiplied with b, and then you will have

$\left.\begin{array}{ll} d+b & \text{the Numerator,} \\ \hline dd+bd+bb & \text{the Denominator,} \end{array}\right\}$ of the Fraction required.

But if after all the Means used (as above) there cannot be found one common Measure to both the Numerator and Denominator; then is that Fraction in its least Terms already.

Note, These Operations will be understood by a Learner after he hath passed thro' *Multiplication*, and *Division* of Fractions.

Sect. 5. ADDITION *and* SUBSTRACTION *of Fractional* QUANTITIES.

THE given Fractions being of one Denomination, or if they are not, make them so, per Sect. 4. Then,

RULE.

Add or substract their Numerators, as Occasion requires, and to their Sum or Difference, subscribe the common Denominator: as in Vulgar Fractions.

Examples in ADDITION.

	1	$\frac{bb}{c}$	$\frac{a+b}{d}$	$\frac{2a-b}{d+c}$	$\frac{a-b+d}{d+a}$
	2	$\frac{aa}{c}$	$\frac{2a+c}{d}$	$\frac{2b-a}{d+c}$	$\frac{a+b-d}{d+a}$
1+2	3	$\frac{bb+aa}{c}$	$\frac{3a+b+c}{d}$	$\frac{a+b}{d+c}$	$\frac{2a}{d+a}$

Examples

Examples in SUBSTRACTION.

	1	$\frac{bb+aa}{c}$	$\frac{a+b}{d+c}$	$\frac{3a+b+c}{d}$	$\frac{2b}{d+a}$
	2	$\frac{bb}{c}$	$\frac{2b-c}{d+c}$	$\frac{2a+c}{d}$	$\frac{a+b-d}{d+a}$
1—2	3	$\frac{aa}{c}$	$\frac{2a-b}{d+c}$	$\frac{a+b}{d}$	$\frac{b-a+d}{d+a}$

Sect. 6. MULTIPLICATION *of Fractional Quantities.*

FIrst prepare mixed Quantities (if there be any) by making them improper Fractions, and whole Quantities by subscribing an Unit under them; as per Sect. 3. Then,

RULE.

Multiply the Numerators together for a new Numerator, and the Denominators together for a new Denominator; as in Vulgar Fractions.

Thus	1	$\frac{ab}{c}$	$\frac{3a-2b}{2d+c}$
	2	$\frac{d}{f}$	$\frac{4a+2b}{d}$
1×2	3	$\frac{abd}{cf}$	$\frac{12aa-2ab-4bb}{2dd+dc}$

Suppose it were required to multiply $2a+\frac{b}{c}-25$ with $3b+4c$. These prepared for the Work (per Sect. 3.) will stand

Thus	1	$\frac{2ac+b-25c}{c}$
	2	$\frac{3b+4c}{1}$
2×2	3	$\frac{6bac+3bb-75bc+8acc+4bc-100cc}{c}$
or	4	$6ba-71b+8ac-100c+\frac{3bb}{c}$ per. Sect. 4.

N. B.

N. B. Any Fraction is multiplied with its Denominator by casting off, or taking the Denominator away. Thus $\frac{b}{a} \times a$ gives b. For $\frac{b}{a} \times \frac{a}{1} = \frac{ba}{a} = b$, &c.

Sect. 7. Division *of Fractional Quantities.*

THE Fractional Quantities being prepared, as directed in the last Section. Then,

RULE.

Multiply the Numerator of the Dividend, into the Denominator of the Divisor, for a new Numerator; and multiply the other two together for a new Denominator; as in Vulgar Fractions.

EXAMPLES.

Let $\frac{abd}{cf}$ be divided by $\frac{ab}{c}$, the Work will stand thus,

$$\frac{ab}{c}\Big)\frac{abd}{cf}\Big(\frac{abdc}{abcf} = \frac{d}{f}$$ per Sect. 4.

Or thus				
	1	$\frac{abd}{cf}$	$\frac{a+b}{d}$	$\frac{aaa-bbb}{a+b}$
	2	$\frac{ab}{c}$	$\frac{c-b}{a}$	$\frac{aa-ab+bb}{c}$
$1 \div 2$	3	$\frac{d}{f}$	$\frac{aa+ba}{dc-db}$	$\frac{aaac-bbbc}{aaa+bbb}$

Suppose it were required to divide $aa + \frac{3abb}{a+4b}$ by $a+b$. The Work prepared will stand thus,

$$\frac{a+b}{1}\Big)\frac{aaa+4aab+3abb}{a+4b}\Big(\frac{aaa+4aab+3abb}{aa+5ba+4bb}.$$ But

$$\frac{aaa+4aab+3abb}{aa+5ba+4bb} = \frac{aa+3b}{a+4b}$$ (per Sect. 4.)

When Fractions are of one Denomination, cast off the Denominators, and divide the Numerators. Thus, if $\frac{ab^3}{c}$ were to be divided by $\frac{bb}{c}$ it will be $bb)\ ab^3\ (ab$ the Quotient required.

For $\frac{bb}{c}\Big)\frac{ab^3}{c}\Big(\frac{ab^3c}{bbc}$ But $\frac{ab^3}{bbc} = ab$ (per Sect. 4.)

Again, suppose it were required to divide $\frac{a^3 - abb}{c - d}$ by $\frac{aa + 2ab + bb}{c - d}$ Casting off $c - d$ in both it will be

$aa + 2ab + bb\Big)\ aaa - abb\ \Big(\frac{aa - ba}{a + b}$, &c.

Sect. 8. Involution *of Fractional Quantities.*

RULE.

Involve the Number into itself for a new Numerator, and the Denominator into itself for a new Denominator; each as often as the Power requires.

Thus	1	$\frac{b}{a}$	$\frac{3\,bc}{2\,ad}$	$\frac{b+d}{a-c}$
$2Q^4$	2	$\frac{bb}{aa}$	$\frac{9\,bbcc}{4\,aadd}$	$\frac{bb+2bd+dd}{aa-2ac+cc}$
$1Q^3$	3	$\frac{bbb}{aaa}$	$\frac{27\,bbbccc}{8\,aaaddd}$	$\frac{bbb+3bbd+3bdd+ddd}{aaa-3aac+3acc-ccc}$

Sect. 9. Evolution *of Fractional Quantities.*

If the Numerator and Denominator of the Fraction have each of them such a Root as is required (which very rarely happens) then involve them; and their respective Roots will be the Numerator and Denominator of the new Fraction required.

Thus	1	$\frac{9\,aabb}{4\,dd}$	$\frac{aa+2ab+bb}{aa-2ab+bb}$
$1w^2$	2	$\frac{3\,ab}{2d}$	$\frac{a+b}{a-b}$

Again,	1	$\frac{27\,aaabbb}{8\,ddd}$	$\frac{aaa+3aab+3abb+bbb}{aaa-3aab+3abb-bbb}$
$1w^3$	2	$\frac{3\,ab}{2\,d}$	$\frac{a+b}{a-b}$

Sometimes it so falls out, that the Numerator may have such a Root as required, when the Denominator hath not; or the Denominator

minator may have ſuch a Root, when the Numerator hath not. In thoſe Caſes the Operations may be ſet down.

Thus	1	$\frac{aabb}{ddd}$	$\frac{aaa+4bb-dd}{aa+2ab+bb}$
$1\omega^2$	2	$\frac{ab}{\sqrt{ddd}}$	$\frac{\sqrt{aaa+4bb-dd}}{a+b}$

But when neither the Numerator, nor the Denominator have juſt ſuch a Root as is required, prefix the radical Sign of the Root to the Fraction; and then it becomes a Surd; as in the laſt Step, which brings me to the Buſineſs of managing Surds.

CHAP. IV.

Of Surd Quantities.

THE whole Doctrine of Surds (as they call it) were it fully handled, would require a very large Explanation (to render it but tolerably intelligible); even enough to fill a Treatiſe itſelf, if all the various Explanations that may be of Uſe to make it eaſy ſhould be inſerted; without which it is very intricate and troubleſome for a Learner to underſtand. But now theſe tedious Reductions of Surds, which were heretofore thought uſeful to fit Equations for ſuch a Solution, as was then underſtood, are wholly laid aſide as uſeleſs: Since the new Methods of reſolving all Sorts of Equations render their Solutions equally eaſy, altho' their Powers are never ſo high. Nay, even ſince the true Uſe of Decimal *Arithmetick* hath been well underſtood, the Buſineſs of Surd Numbers has been managed that Way; as appears by ſeveral Inſtances of that Kind in Dr. *Wallis*'s *Hiſtory of Algebra*, from *Page* 23, *to* 29.

I ſhall therefore, for Brevity ſake, paſs over thoſe tedious Reductions, and only ſhew the young *Algebraiſt* how to deal with ſuch Surd Quantities as may ariſe in the Solution of hard Queſtions.

Sect. 1. Addition *and* Substraction *of Surd Quantities.*

Caſe 1. WHen the Surd Quantities are Homogeneal, (*viz.* are alike) add, or ſubſtract the rational Part, if they

are joined to any, and to their Sum, or Difference, adjoin the Irrational or Surd.

Examples in ADDITION.

	1	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
1+2	3	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$

	1	$4d\sqrt[3]{aa}$	$b+\sqrt[3]{aa-cc}$	$bc : \sqrt[5]{aa+d}$
	2	$d\sqrt[3]{aa}$	$c-\sqrt[3]{aa-cc}$	$3bc : \sqrt[5]{aa+d}$
1+2	3	$5d\sqrt[3]{aa}$	$b+c$	$4bc : \sqrt[5]{aa+d}$

Examples in SUBSTRACTION.

	1	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
1−2	3	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$

	1	$5d : \sqrt[3]{aa}$	$b+c$	$4bc : \sqrt[5]{aa+d}$
	2	$4d : \sqrt[3]{aa}$	$c-\sqrt[3]{aa-cc}$	$3bc : \sqrt[5]{aa+d}$
1−2	3	$d : \sqrt[3]{aa}$	$b+\sqrt[3]{aa-cc}$	$bc : \sqrt[5]{aa+d}$

Case 2. When the Surd Quantities are Heterogeneal, (*viz.* their Indices are unlike) they are only to be added or substracted by their Signs, *viz.* + or —. And from thence will arise Surds either Binomial, or Residual.

Examples in ADDITION.

	1	$\sqrt{bc}$	$4d\sqrt{a}$	$\sqrt[3]{ac-ba}$
	2	$\sqrt{ba}$	$3b\sqrt{ac}$	$\sqrt{ac+ba}$
1+2	3	$\sqrt{bc} : +\sqrt{ba}$	$4d\sqrt{a} : +3b\sqrt{ac}$	$\sqrt[3]{ac-ba} : +\sqrt{ac+ba}$

Examples in SUBSTRACTION.

	1	$\sqrt{bc}$	$b-d\sqrt{aaa+ca}$
	2	$\sqrt{ba}$	$d-2a\sqrt{bd+dd}$
1−2	3	$\sqrt{bc}-\sqrt{ba}$	$b-d\sqrt{aaa+ca} : -d+2a\sqrt{bd+dd}$

Sect.

Sect. 2. MULTIPLICATION *of Surd Quantities.*

Case 1. WHEN the Quantities are pure Surds of the same Kind; multiply them together, and to their Product prefix their radical Sign.

EXAMPLES.

	1	$\sqrt{b}$	$\sqrt{ba+da}$	$\sqrt{aa+bb}$
	2	$\sqrt{a}$	$\sqrt{ca}$	$\sqrt{aa-bb}$
1×2	3	$\sqrt{ba}$	$\sqrt{bcaa+dcaa}$	$\sqrt{aaaa-bbbb}$

Case 2. If Surd Quantities of the same Kind (as before) are joined to rational Quantities, then multiply the rational into the rational; and the Surd into the Surd, and join their Products together.

EXAMPLES.

	1	$d\sqrt{bc}$	$5cd\sqrt{ba+da}$	$15\sqrt{ab}$
	2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
1×2	3	$3db\sqrt{bca}$	$15cda\sqrt{bcaa+dcaa}$	$75\sqrt{abd}$

Sect. 3. DIVISION *of Surd Quantities.*

Case 1. WHEN the Quantities are pure Surds of the same Kind, and can be divided off, (*viz.* without leaving a Remainder) divide them, and to their Quotient prefix their radical Sign.

EXAMPLES.

	1	$\sqrt{ba}$	$\sqrt{bcaa+dcaa}$	$\sqrt{aaaa-bbbb}$
	2	$\sqrt{b}$	$\sqrt{ca}$	$\sqrt{aa-bb}$
$1\div 2$	3	$\sqrt{a}$	$\sqrt{ba+da}$	$\sqrt{aa+bb}$

Case 2. If Surd Quantities, of the same Kind, are joined to rational Quantities; then divide the rational by the rational, if it can be, and to their Quotient join the Quotient of the Surd divided by the Surd with its first radical Sign.

EXAMPLES.

	1	$3db\sqrt{bca}$	$15cda\sqrt{bcaa+dcaa}$	$75\sqrt{abd}$
	2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
$1\div 2$	3	$d\sqrt{bc}$	$5cd\sqrt{ba+da}$	$15\sqrt{ab}$

Note, If any Square be divided by its Root, the Quotient will be its Root.

EXAMPLES.

	1	a	$bb+2bc+cc$	$aaaa-2bbaa+bbbb$
	2	$\sqrt{a}$	$\sqrt{bb+2bc+cc}$	$\sqrt{a^4-2bbaa+b^4}$
$1 \div 2$	3	$\sqrt{a}$	$\sqrt{bb+2bc+cc}$	$\sqrt{a^4-2bbaa+b^4}$

Sect. 4. Involution *of Surd Quantities.*

Case 1. WHEN the Surds are not joined to rational Quantities they are involved to the same Height as their Index denotes, by only taking away their radical Sign.

EXAMPLES.

	1	$\sqrt{a}$	$\sqrt{bca}$	$\sqrt{aa-bb}$	$\sqrt{5a-da}$
$1 \circledcirc^{2}$	2	a	bca	$aa-bb$	$5a-da$

Case 2. When the Surds are joined to rational Quantities; involve the rational Quantities to the same Height as the Index of the Surd denotes; then multiply those involved Quantities into the Surd Quantities, after their radical Sign is taken away, as before.

EXAMPLES.

	1	$b\sqrt{a}$	$5d\sqrt{ca}$	$3b\sqrt{aa-dd}$
$1 \circledcirc^{2}$	2	bba	$25\,ddca$	$9bbaa-9bbdd$

	1	$a:\sqrt[3]{bc}$	$3d:\sqrt[3]{aa+bb}$	$da:\sqrt[3]{b}$
$1 \circledcirc^{3}$	2	$aaabc$	$27dddaa+27dddbb$	$dddaaab$

The Reason of only taking away the radical Sign, as in *Case* 1. is easily conceived, if you consider that any Root being involved into itself, produces a Square, &c. And from thence the Reason of those Operations performed by the second Case may be thus stated.

Suppose $b\sqrt{a}=x$. Then $\sqrt{a}=\frac{x}{b}$ *per Axiom* 4. and both Sides of the Equation being equally involved, it will be $a=\frac{xx}{bb}$. Then multiplying both Sides of the Equation into bb, it becomes $bba=xx$ *per Axiom* 3. Which was to be proved.

Again,

Again, Let $5d\sqrt{ca} = x$: Then $\sqrt{ca} = \frac{x}{5d}$, and $ca = \frac{xx}{25dd}$

Alſo from hence it will be eaſy to deduce the Reaſon of multiplying Surd Quantities, according to both the *Caſes*. For

Suppoſe	1	$\sqrt{b} = x$ } *Example* 1. *Caſe* 1.
	2	$\sqrt{a} = x$
1 ⊙ 2	3	$b = xx$
2 ⊙ 2	4	$a = xx$
3 × 4	5	$ba = xxxx$. *per Axiom* 2.
5 ш 2	6	$\sqrt{ba} = xx$, which was to be proved.

Let	1	$d\sqrt{bc} = x$ } *Example* 1. *Caſe* 2.
	2	$3b\sqrt{a} = x$
$1 \div d$	3	$\sqrt{bc} = \frac{x}{d}$
$2 \div 3b$	4	$\sqrt{a} = \frac{x}{3b}$
4×3	5	$\sqrt{abc} = \frac{xx}{3bd}$ from what is proved above.
$5 \times 3bd$	6	$3bd\sqrt{bca} = xx$, &c. for the reſt.

Diviſion being the Converſe to *Multiplication*, needs no other Proof.

CHAP. V.

Concerning the Nature of EQUATIONS *and how to prepare them for a* SOLUTION.

WHEN any Problem or Queſtion is propoſed to be analytically reſolved; it is very requiſite that the true Deſign or Meaning thereof, be fully and clearly comprehended (in all its Parts) that ſo it may be truly abſtracted from ſuch ambiguous Words as Queſtions of this Kind are often diſguiſed with, otherwiſe it will be very difficult, if not impoſſible, to ſtate the Queſtion right in its ſubſtituted Letters, and ever to bring it to an Equation by ſuch various Methods of ordering thoſe Letters as the Nature of the Queſtions may require.

Now

Now the Knowledge of this difficult Part of the Work is only to be obtained by Practice, and a careful minding the Solution of such leading Questions as are in themselves very easy. And for that Reason I have inserted a Collection of several Questions; wherein there is great Variety.

Having got so clear an Understanding of the Question proposed, as to place down all the Quantities concerned in their due Order, *viz.* all the substituted Letters, in such Order as their Nature requires; the next thing must be to consider whether it be limited or not. That is, whether it admits of more Answers than one. And to discover that, observe the two following Rules.

RULE 1.

When the Number of the Quantities sought exceed the Number of the given Equations, the Question is capable of innumerable Answers.

EXAMPLE.

Suppose a Question were proposed thus; there are three such Numbers, that if the first be added to the second, their Sum will be 22. And if the second be added to the third, their Sum will be 46. What are those Numbers?

Let the three Numbers be represented by three Letters, thus, call the first a, the second e, and the third y.

Then $\left\{ \begin{array}{l} a+e=22 \\ e+y=46 \end{array} \right\}$ according to the Question.

Here the Number of Quantities sought are three; a, e, y, and the Number of the given Equations are but two. Therefore this Question is not limited, but admits of various Answers; because for any one of those three Letters you may take any Number at Pleasure, that is less than 22. Which with a little Consideration will be very easy to conceive.

RULE 2.

When the Number of the given Equations (not depending upon one another) are just as many as the Number of the Quantities sought; then is the Question truly limited, viz. *each Quantity sought hath but one single Value.*

As for Instance, let the aforesaid Question be proposed thus. There are three Numbers (a, e, and y, as before) if the first be added to the second, their Sum will be 22; if the second be added

to

to the third, their Sum will be 46; and if the first be added to the third, their Sum will be 36. What are the Numbers? That is, $a+e=22$. $e+y=46$. and $a+y=36$. Now the Question is perfectly limited, each single Quantity having but one single Value, to wit, $a=6$, $e=16$, and $y=30$.

N.B. If the Number of the given Equations exceeds the Number of the Quantities sought; they not only limit the Question, but oftentimes render it impossible, by being proposed inconsistent one to another.

Having truly stated the Question in its substituted Letters, and found it limited to one Answer (or at least so bounded as to have a certain determinate Number of Answers) then let all those substituted Letters be so ordered or compared together, either by adding, substracting, multiplying, or dividing them, &c. according as the Nature of the Question requires, 'till all the unknown Quantities except one, are cast off or vanished; but therein great Care must to be taken to keep them to an exact Equality; and when that unknown Quantity, or some Power of it (as Square, Cube, &c.) is found equal to those that are known; then the Question is said to be brought to an Equation, and consequently to a Solution, *viz.* fitted for an Answer.

But no particular Rules can be prescribed for the casting off, or getting away Quantities out of an Equation; that Part of the Art is only to be obtained by Care and Practice. And when that is done, it generally happens so, that the unknown Quantity which is retained in the Equation, is so mingled and entangled with those that are known; that it often requires some Trouble and Skill to bring it (or its Powers, &c.) to one Side of the Equation, and those that are known to the other side; (still keeping them to a just Equality) which the ingenious Mr. *Scooten* in his *Principia Matheseos Universalis*, calls Reduction of Equations.

The Business of reducing Equations *(as of most, if not all Algebraic Operations)* is grounded and depends upon a right Application of the five *Axioms* proposed in Page 146, and therefore, if those *Axioms* be well understood, the Reason of such Operations must needs appear very plain, and the Work be easily performed; as in the following *Sections*.

Sect. 1. *Of* Reduction *by* Addition.

REDUCTION by *Addition* is grounded upon *Axiom* 1. and is only the transposing (*viz.* the removing) of any negative Quantity from either Side of an Equation to the other Side, with the Sign + before it; as in these

EXAMPLES.

Suppose	1	$a-b=d$
Then	2	$a=d+b$
For	3	$b=b$
1+3	4	$a=d+b$

Again

Let	1	$aa-d=c-aa$
$1+d$	2	$aa=c-aa+d$
$2+aa$	3	$2aa=c+d$

Let	1	$3a-4=6-a$
$1+\overline{4}$	2	$3a=6+4-a$
$2+a$	3	$4a=6+4=10$

Note, When any absolute Number is registered in the Margin, you must draw a Line over it, to distinguish it from the other Numbers. As $\overline{4}$ in the 2d Step of this Example.

Let	1	$aa-dc-b=dd-2ba$
$1+b$	2	$aa-dc=dd-2ba+b$
$2+dc$	3	$aa=dd-2ba+b+dc$
$3+2ba$	4	$aa+2ba=dd+b+dc$

Suppose	1	$2da-d=cc-3baa-aaa$
$1+aaa$	2	$aaa+2da-d=cc-3baa$
$2+3baa$	3	$aaa+3baa+2da-d=cc$
$3+d$	4	$aaa+3baa+2da=cc+d$, &c.

Sect. 2. *Of* Reduction *by* Substraction.

REDUCTION by *Substraction* is grounded upon *Axiom* 2. and is performed by transposing (or removing) any affirmative Quantity from either Side of the Equation, to the other Side, with the Sign — before it; as in these

EXAMPLES.

Suppose	1	$a+b=d$
And	2	$b=b$
1—2	3	$a=d-b$

Let	1	$3a+4=6+a$
$1-a$	2	$2a+4=6$
$2-\overline{4}$	3	$2a=6-4=2$

Suppose	1	$aa+dc+b=dd+2ba$
$1-2ba$	2	$aa-2ba+dc+b=dd$
$2-dc$	3	$aa-2ba+b=dd-dc$
$3-b$	4	$aa-2ba=dd-dc-b$

Let

Let	1	$aaa+d=cc+3baa+2da$
$1-3baa$	2	$aaa-3baa+d=cc+2da$
$2-2da$	3	$aaa-3baa-2da+d=cc$
$3-d$	4	$aaa-3baa-2da=cc-d$

Sect. 3. *Of* REDUCTION *by* MULTIPLICATION.

FRACTIONAL Quantities, in any Equation, are brought into whole Quantities by multiplying every Term in the Equation with the Denominators of the Fractions, *per Axiom* 3; as in these

EXAMPLES.

Suppose	1	$\frac{a}{5}=6$
Then	2	$a=6\times5=30$. For $\frac{a}{5}\times5=\frac{5a}{5}=a$.

Let	1	$3a=\frac{dc}{2b}$
$1\times2b$	2	$6ba=dc$

Suppose	1	$a=\frac{dd}{a-b}$
$1\times\overline{a-b}$	2	$aa-ba=dd$

Suppose	1	$\frac{aa}{b}+c+f=\frac{dx}{a}$
$1\times b$	2	$aa+bc+bf=\frac{dxb}{a}$
$2\times a$	3	$aaa+bca+bfa=dxb$

Suppose	1	$\frac{aaa}{aa-bb}=\frac{ba-bb}{a+b}$
$1\times\overline{aa-bb}$	2	$aaa=\frac{baaa-bbaa-bbba+bbbb}{a+b}$
$2\times\overline{a+b}$	3	$aaaa+baaa=baaa-bbaa-bbba+bbbb$

Sect. 4. *Of* REDUCTION *by* DIVISION.

WHEN any Quantity (either known or unknown) is in every Term of an Equation, if the whole Equation be divided by that Quantity, it will be reduced into lower Terms, *per Axiom* 4, as in these following *Examples*.

EXAMPLES.

$$\begin{array}{r|c|l}
\text{Suppose} & 1 & baa+bca=bcd \\
1 \div b & 2 & aa+ca=cd
\end{array}
\qquad
\begin{array}{r|c|l}
\text{Let} & 1 & aa=7a \\
1 \div 1a & 2 & a=7
\end{array}$$

$$\begin{array}{r|c|l}
\text{Let} & 1 & ffaa+ffcaa-ffa=ffda+ffdda \\
1 \div ff & 2 & aa+caa-a=da+dda \\
2 \div a & 3 & a+ca-1=d+dd
\end{array}$$

Or when the unknown Quantity is multiplied (*viz.* joined) with any that is known; let the whole Equation be divided by the known Quantity, that so the unknown may be cleared; as in these

EXAMPLES.

$$\begin{array}{r|c|l}
\text{Suppose} & 1 & ba-ca=d \\
1 \div \overline{b-c} & 2 & a=\dfrac{d}{b-c}
\end{array}
\qquad
\begin{array}{r|c|l}
\text{Let} & 1 & caa-daa=cd-dd \\
1 \div \overline{c-a} & 2 & aa=\dfrac{cd-dd}{c-d}=d.
\end{array}$$

$$\begin{array}{r|c|l}
\text{Suppose} & 1 & bbaaa-2bbaa=bda+cba \\
1 \div ba & 2 & baa-2ba=d+c \\
2 \div b & 3 & aa-2a=\dfrac{d+c}{b}
\end{array}$$

$$\begin{array}{r|c|l}
\text{Let} & 1 & 49daa+42aa=7bca+21ca \\
1 \div 7 & 2 & 7daa+6aa=bca+3ca \\
2 \div a & 3 & 7da+6a=bc+3c \\
3 \div d & 4 & a=\dfrac{bc+3c}{7d+6}
\end{array}$$

Sect. 5. *Of* REDUCTION *by* INVOLUTION.

WHEN there happens to be an Equation, between any homogeneal or like Surds, take away the radical Signs from the Quantities, and they will become rational; as in these

EXAMPLES.

$$\begin{array}{r|c|l}
\text{Suppose} & 1 & \sqrt{a}=\sqrt{d+c} \\
1 \circledS^{2} & 2 & a=d+c
\end{array}
\qquad
\begin{array}{r|c|l}
\text{Let} & 1 & \sqrt[3]{aa}=\sqrt[3]{db+bc} \\
1 \circledS^{3} & 2 & aa=db+bc
\end{array}
\Big\}\ \textit{per Sect. 4. Chap. 3.}$$

Or if one Side of the Equation consists of Surd Quantities, and the other Side be rational, then involve the rational Quantities to the

the ſame Power (or Height) with the Index of the Surd, and take away the radical Sign; as in theſe

EXAMPLES.

Let	1	$\sqrt{a}=6$	Suppoſe	1	$\sqrt{a}=b+c$
1 ⓖ^{2}	2	$a=36$	1 ⓖ^{2}	2	$a=bb+2bc+cc$

Suppoſe	1	$\sqrt[3]{aa-ba}=d$	Let	1	$\sqrt[5]{aa}=7$
1 ⓖ^{3}	2	$aa-ba=ddd$	1 ⓖ^{2}	2	$aa=16807$.

Sect. 6. *Of* Reduction *by* Evolution.

WHEN any ſingle Power of the unknown Quantity is on one Side of an Equation; evolve both Sides of the Equation, according as the Index of that Power denotes, and their Roots will be equal; as in theſe

EXAMPLES.

Suppoſe	1	$aa=36$	Let	1	$aaa=27$
1 ω^2	2	$a=\sqrt{36}=6$	1 ω^3	2	$a=\sqrt[3]{27}=3$, &c.

Suppoſe	1	$aa=bb-dd$	Let	1	$aaa=b^3+3bbc+3bcc+c^3$
1 ω^2	2	$a=\sqrt{bb-dd}$	1 ω^3	2	$a=b+c$

Or if any compound Power of the unknown Quantity be on one Side of the Equation (that hath a true Root of its kind) evolve both Sides of the Equation, and it will be depreſſed into lower Terms; as in theſe

EXAMPLES.

Suppoſe	1	$aa+2ba+bb=dd$	$aa-2ba+bb=ddee$
1 ω^2	2	$a+b=d$	$a-b=\pm de$

Here follow a few Examples of clearing Equations, wherein all the foregoing Reductions are promiſcuouſly uſed, as Occaſion requires.

EXAMPLE 1.

Suppoſe	1	$\dfrac{aa+c-d}{4}=\dfrac{g-aa}{b}$ what is $a=$ to?
1 × 4	2	$aa+c-d=\dfrac{4g-4aa}{b}$

2 × b

$2 \times b$	3	$baa + bc - bd = 4g - 4aa$
$3 + 4aa$	4	$baa + 4aa + bc - bd = 4g$
$4 + bd$	5	$baa + 4aa + bc = 4g + bd$
$5 - bc$	6	$baa + 4aa = 4g + bd - bc$
$6 \div \overline{b+4}$	7	$aa = \frac{4g + bd - bc}{b+4}$
$7\ \text{ɯ}^2$	8	$a = \sqrt{\frac{4g+bd-bc}{b+4}}$ as was required.

EXAMPLE 2.

Suppose	1	$\frac{b+354}{a} = \frac{3a}{354-a}$ what is the Value of a?
$1 \times a$	2	$a + 354 = \frac{3aa}{354-a}$
$2 \times \overline{354-a}$	3	$125316 - aa = 3aa$
$3 + aa$	4	$4aa = 125316$
$4 \div 4$	5	$aa = 31329$
$5\ \text{ɯ}^2$	6	$a = \sqrt{31329} = 177$, the Value of a required.

EXAMPLE 3.

Suppose	1	$\sqrt{\frac{aa+3bb}{4}} - \sqrt{\frac{aa-3bb}{4}} = \sqrt{\frac{baa}{c}} : a = ?$
1 ⓢ^{2}	2	$\left\{ \begin{array}{l} \frac{aa+3bb}{4} - 2\sqrt{\frac{aa+3bb}{4}} \times \sqrt{\frac{aa-3bb}{4}} \\ : + \frac{aa-3bb}{4} = \frac{baa}{c} \end{array} \right.$
That is	3	$\frac{aa}{2} - \sqrt{\frac{a^4 - 9b^4}{4}} = \frac{bba}{c}$
For		$\frac{aa+3bb}{4} + \frac{aa-3bb}{4} = \frac{2aa}{4} = \frac{aa}{2}$
And		$2\sqrt{\frac{aa+3bb}{4}} = \sqrt{\frac{4aa+12bb}{4}} = \sqrt{aa+3bb}$
Then		$\sqrt{aa+3bb} \times \sqrt{\frac{aa-3bb}{4}} = \sqrt{\frac{a^4-9b^4}{4}}$
$3 + \sqrt{}$ &c.	4	$\frac{aa}{2} = \frac{baa}{c} + \sqrt{\frac{a^4-9b^4}{4}}$

4 —

$4 - \frac{baa}{c}$	5	$\frac{aa}{2} - \frac{baa}{c} = \sqrt{\frac{a^4 - 9b^4}{4}}$
5 ⓢ 2	6	$\frac{a^4}{4} - \frac{ba^4}{c} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4}$
$6 + \frac{ba^4}{c}$	7	$\frac{a^4}{4} + \frac{bba^4}{cc} = \frac{a^4 - 9b^4}{4} + \frac{ba^4}{c}$
$7 \pm$	8	$\frac{bba^4}{cc} + \frac{9b^4}{4} = \frac{ba^4}{c}$
$8 \div b$	9	$\frac{ba^4}{cc} + \frac{9b^3}{4} = \frac{a^4}{c}$
$9 \times cc$	10	$ba^4 + \frac{9ccb^3}{4} = ca^4$
10×4	11	$4ba^4 + 9ccb^3 = 4ca^4$
$11 - 4ba^4$	12	$9ccb^3 = 4ca^4 - 4ba^4$
$12 \div$	13	$aaaa = \frac{9ccb^3}{4c - 4b}$
For		$\overline{4c - 4b} \times a^4 = 4ca^4 - 4ba^4$
13 ɯ 2	14	$aa = \sqrt{\frac{9ccb^3}{4c - 4b}}$
14 ɯ 2	15	$a = \sqrt{:}\sqrt{\frac{9ccb^3}{4c - 4b}}$, as was required.

By Help of these Reductions (properly applied) the unknown Quantity (a) or its Powers, are cleared and brought to one Side of an Equation; and if the unknown Quantity (a) chance to be equal to those that are known, the Question is answered: as in the first *Example* of *Sect.* 1, and 2. Or if any single Power of the unknown Quantity (a) is found equal to those that are known, then the respective Root of the known Quantities is the Answer; as in the first four *Examples* of *Sect.* 6, &c.

But when the Powers of the unknown Quantities are either mixed with their Root, as $aa + ba = dd$, &c. or do consist of different Powers, as $aaa + baa = dd$, &c. Then they are called Affected, or Adfected Equations, which require other Methods to resolve them; *viz.* to find out the Value of (a) as shall be shewed further on.

CHAP.

CHAP. VI.

Of PROPORTIONAL QUANTITIES; *both* ARITHMETICAL, GEOMETRICAL, *and* MUSICAL.

WHat hath been ſaid of Numbers in *Arithmetical Progreſſion*, Chap. 6. Part 1. may be eaſily applied to any Series of Homogeneal or like Quantities.

Sect. 1. *Of* QUANTITIES in ARITHMETICAL PROGRESSION.

THOSE Quantities are ſaid to be in the moſt ſimple or natural Progreſſion, that begin their Series of Increaſe or Decreaſe with a Cypher :

Thus $\begin{cases} 0 : a : 2a : 3a : 4a : 5a : 6a : \text{\&c. increasing.} \\ 0 : -a : -2a : -3a : -4a : -5a : -6a : \text{\&c. decreasing.} \end{cases}$

Or Univerſally, putting a the firſt Term in the Progreſſion, and e the common Exceſs or Difference.

Then $\begin{cases} a : a+e : a+2e : a+3e : a+4e : a+5e : a+6e : \text{\&c.} \\ a : a-e : a-2e : a-3e : a-4e : a-5e : a-6e : \text{\&c.} \end{cases}$

In the firſt of theſe Series it is evident, that if there be but three Terms; the Sum of the Extreams will be double to the Mean.

As in theſe, $o : a : 2a$: or, $a : 2a : 3a$: or, $2a : 3a : 4a$, &c. *viz.* $2a : +o=a+a$: or, $a+3a=2a+2a$, &c.

Alſo, in the ſecond Series, either increaſing or decreaſing, it is evident, that if the Terms be $a : a+e : a+2e$, &c. increaſing; then $a+a+2e$, *viz.* $2a+2e$ the Sum of the Extreams, is double to $a+e$ the Mean, or if they be $a : a-e : a-2e$, &c. decreaſing, then $a+a-2e$: *viz.* $2a-2e$ the Sum of the Extreams, is double $a-e$ the Mean. And ſo it will be in any other three of the Terms. Secondly, if there are four Terms; then the Sum of the two Extreams, will be equal to the Sum of the two Means; as in theſe, $a : a+e : a+2e : a+3e$, in the Series increaſing; here, $a+a+3e=a+e+a+2e$.

Alſo in theſe, $a : a-e : a-2e : a-3e$ in the Series decreaſing; here $a+a-3e=a-e+a-2e$, &c. in any other four Terms.

Conſequently, If there are never ſo many Terms in the Series, the Sum of the two Extreams will always be equal to the Sum of

of any two Means, that are equally diſtant from thoſe Extreams. As in theſe, $a : a+e : a+2e : a+3e : a+4e : a+5e :$ &c. Here $a+a+5e = a+e+a+4e = a+2e+a+3e$, &c. And if the Number of Terms be odd, the Sum of the two Extreams will be double to the middle Term, &c. as in Corol. 1. Chap. 6. before-mentioned.

CONSECTARY 1.

Whence it follows, (and is very eaſy to conceive) that if the Sum of the two Extreams be multiplied into the Number of all the Terms in the Series, the Product will be double the Sum of all the Series.

Now for the eaſier reſolving ſuch Queſtions as depend upon theſe Progreſſional Quantities.

$$\text{Let}\begin{cases} a = \text{the firſt Term, as before.} \\ y = \text{the laſt Term.} \\ e = \text{the common Exceſs, \&c. as before.} \\ N = \text{the Number of all the Terms.} \\ S. = \text{the Sum of all the Series, viz. of all the Terms.} \end{cases}$$

Then will $\overline{a+y} \times N = 2S$, by the precedent Conſectary: that is, $Na + Ny = 2S$. Conſequently $\frac{Na+Ny}{2} = S$ the Sum of all the Series, be the Terms never ſo many. Thirdly, In theſe Series it is eaſy to perceive, that the common Difference (e) is ſo often added to the laſt Term of the Series; as are the Number of Terms except the firſt; that is, the firſt Term (a) hath no Difference added to it, but the laſt Term hath ſo many Times (e) added to it, as it is diſtant from the firſt.

Conſequently, the Difference betwixt the two Extreams, is only the common Difference (e) multiplied into the Number of all the Terms leſs Unity or 1. That is, $\overline{N-1} \times e = y-a$, the Difference betwixt the two Extreams, *viz.* $Ne - e = y - a$.

CONSECTARY 2.

Whence it follows, that if the Difference betwixt the two Extreams be divided by the Number of Terms leſs 1, the Quotient will be the common Difference of the Series.

To wit, $\frac{y-a}{N-1} = e$

Now by the Help of these two Consectaries, if any three of the aforesaid five Parts (*viz. a. y. e. N. S.*) be given; the other two may be easily found.

Thus,	1	$\frac{Na+Ny}{2}=S$ } as before.
And	2	$\frac{y-a}{N-1}=e$ }
$2\times\overline{N-1}$	3	$y-a=Ne-e$
$3+e$	4	$y-a+e=Ne$
$4\div e$	5	$\frac{y-a+e}{e}=N$, the Number of Terms.
1×2	6	$Na+Ny=2S$
$6-Na$	7	$Ny=2S-Na$
$7\div N$	8	$\frac{2S-Na}{N}=y$, the last Term.
$6-yN$	9	$Na=2S-Ny$
$9\div N$	10	$\frac{2S-Ny}{N}=a$, the first Term.
$6\div\overline{a+y}$	11	$\frac{2S}{a+y}=N$, the Number of Terms.
5, and 11	12	$\frac{y-a+e}{e}=\frac{2S}{a+y}$ *per Axiom* 5.
$12\times\overline{a+y}$	13	$\frac{yy-aa}{e}+a+y=2S$
$13\div 2$	14	$\frac{yy-aa}{2e}+\frac{a+y}{2}=2S$, the Sum of all the Series.
$14\times 2e$	15	$yy-aa+ae+ye=2Se$
$15-ae$	16	$yy-aa+ye=2Se-ae$
$17\div$ $2S-a-y$	18	$\frac{yy-aa}{2S-a-y}=e$, the common Difference.
$16-ye$	17	$yy-aa=2Se-ae-ye$
$3+a$	19	$Ne-e+a=y$, the last Term.
$19+e$	20	$Ne+a=y+e$
$20-Ne$	21	$y+e-Ne=a$, the first Term. &c.

In like Manner you may proceed to find out any of the five Quantities (*a. e. y. N. S.*) otherwise, *viz.* by varying or comparing those Equations one with another, you may produce new Equations

Equations with other Data in them; the which I shall here omit pursuing, and leave them for the Learner's Practice.

Sect. 2. *Of* Quantities *in Geometrical Proportion.*

GEOMETRICAL Proportion continued has been already defined in Sect. 2. Chap. 6. Part I. And what is there said concerning Numbers in $\div\!\!\div$ may easily be applied to any Sort of homogeneal Quantities that are in $\div\!\!\div$.

The most natural and simple Series of geometrical Proportionals, is when it begins with Unity or 1.

As $1 . a . aa . aaa . aaaa . a5 . a^6$, &c. in $\div\!\!\div$
For $1 : a :: a : aa :: aa : aaa :: aaa : aaaa$, &c.

Or $a . b . \frac{bb}{a} . \frac{bbb}{aa} . \frac{bbbb}{aaa} . \frac{b5}{a4}$, &c. are Terms in $\div\!\!\div$

For $a : b :: b : \frac{bb}{a} :: \frac{bb}{a} : \frac{bbb}{aa} :: \frac{bbb}{aa} : \frac{b4}{a3} :: \frac{b4}{a3} : \frac{b5}{a4}$, &c.

That is, when all the middle Terms betwixt the two Extreams are both Consequents and Antecedents, that Series is in geometrical Proportion continued. Therefore in every Series of Quantities in $\div\!\!\div$ all the Terms except the last are Antecedents; and all the Terms except the first are Consequents. But universally putting a the first Term in the Series, and e the Ratio, *viz.* the common Multiplier, or Divisor; then it will be

$a . ae . aee . aeee . aeeee . ae5 . ae^6$, &c. in $\div\!\!\div$

Or $a . \frac{a}{e} . \frac{a}{ee} . \frac{a}{eee} . \frac{a}{eeee} . \frac{a}{e5}$. &c. are $\div\!\!\div$ Decr.

For $a : ae :: ae : \frac{aaee}{a} = aee$, &c.

And $a : \frac{a}{e} :: \frac{a}{e} : \frac{aa}{eee} = \frac{a}{ee} a : \frac{a}{e} :: \frac{a}{ee} : \frac{a}{eee}$, &c.

I. In any of these Series it is evident, that if three Quantities are in $\div\!\!\div$, the Rectangle of the two Extreams will be equal to the Square of the Mean; as in these, $a . ae . aee$, here $a \times aee = ae \times ae, = aaee$. &c.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee}$; here alſo $a \times \frac{a}{ee} = \frac{a}{e} \times \frac{a}{e} = \frac{aa}{ee}$, &c.

II. If four Quantities are in $\div\!\!\div$, the Rectangle of the Extreams will be equal to the Rectangle of the Means.

As in theſe, $a \,.\, ae \,.\, aee \,.\, aeee$; here $a \times ae^3 = ae \times aee$.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee}$; here alſo $a \times \frac{a}{eee} = \frac{a}{e} \times \frac{a}{ee} = \frac{a}{eee}$, &c.

Conſequently, If there are never ſo many Terms in the Series of $\div\!\!\div$, the Rectangle of the Extreams will be equal to the Rectangle of any two Means that are equally diſtant from thoſe Extreams.

As in theſe, $a \,.\, ae \,.\, aee \,.\, aeee \,.\, ae^4 \,.\, ae^5$
viz. $ae^5 \times a = ae^4 \times ae$. Or $ae^5 \times a = aeee \times aee = aae^5$

III. If never ſo many Quantities are in $\div\!\!\div$ it will be, as any one of the Antecedents is to its Conſequents; ſo is the Sum of all the Antecedents, to the Sum of all the Conſequents.

As in theſe $\begin{cases} a \,.\, ae \,.\, aee \,.\, aeee \,.\, aeeee \,.\, ae^5, \text{ \&c. increaſing.} \\ a \,.\, \frac{a}{e} \,.\, \frac{a}{ee} \,.\, \frac{a}{eee} \,.\, \frac{a}{eeee} \,.\, \frac{a}{e^5}, \text{ \&c. decreaſing.} \end{cases}$

$a : ae :: a + ae + aee + ae^3 + ae^4 : ae + aee + ae^3 + ae^4 + ae^5$

Or $a : \frac{a}{e} :: a + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} : \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3} + \frac{a}{e^4} + \frac{a}{e^5}$, viz. $a \times \overline{ae + aee + ae^3 + ae^4 + ae^5} = ae \times \overline{a + ae + aee + ae^3 + ae^4}$.

That is, the Rectangle of the Extreams is equal to the Rectangle of the Means; per Second of this *Sect.*

Note, The Ratio of any Series in $\div\!\!\div$ increaſing is found by dividing any of the Conſequents by its Antecedent.

Thus, $a)\, ae\, (e$ Or $ae)\, aee\, (e$, &c.

But if the Series be decreaſing, then the Ratio is found by dividing any of the Antecedents by its Conſequent.

Thus $\frac{a}{e})\, a\, (e$ Or $\frac{a}{ee})\, \frac{a}{e}\, (e$, &c.

CON-

CONSECTARY.

These Things being premised, such Equations may be deduced from them, as will solve all such Questions as are usually proposed about Quantities in Geometrical Proportion. In Order to that,

let $\begin{cases} a = \text{the first Term.} \\ e = \text{the common Ratio.} \\ y = \text{the last Term.} \\ S = \text{the Sum of all the Terms.} \end{cases}$ (the first two) as before.

Then $S-y=$ the Sum of all the Antecedents.
And $S-a=$ the Sum of all the Consequents.

Analogy.	1	$a : ae :: S-y : S-a$ per III. of this *Sect.*
1 ∵	2	$Sa-aa=aeS-aey$
$2 \div a$	3	$S-a=eS-ey$
$3+ey$	4	$S+ey-a=eS$
$4-S$	5	$ey-a=eS-S$
$5 \div \overline{e-1}$	6	$\dfrac{ye-a}{e-1}=S$, the Sum of all the Series.
$3 \div \overline{S-y}$	7	$\dfrac{S-a}{S-y}=e$, the common Ratio.
$5+a$	8	$ey=eS+a-S$
$8 \div e$	9	$\dfrac{eS+a-S}{e}=y$, the last Term.
$4+a$	10	$S+ey=eS+a$
$10-eS$	11	$S+ey-eS=a$, the first Term.

Note, The ∵ set in the Margin at the second Step, is instead of *ergo*; and imports that the Rectangle of the two Extreams in the first Step, is equal to the Rectangle of the Means. And so for any other Proportion.

Sect. 3. *Of* Harmonical *Proportion.*

HARMONICAL or Musical Proportion is, when of three Quantities (or rather Numbers) the first hath the same Ratio to the third, as the Difference between the first and second, hath to the Difference between the second and third. As in these following.

Suppose a, b, c, in Musical Proportion.

Then	1	$a : c :: b-a : c-b$
1 ∵	2	$cb-ca=ac-ba$

2+

$2+ca$	3	$cb=2ac-ba$
$3\div\overline{2c-b}$	4	$\frac{cb}{2c-b}=a$, the first Term.
$3+ba$	5	$2ac=cb+ba$
$5\div\overline{c+a}$	6	$\frac{2ac}{c+a}=b$, the second Term.
$5-cb$	7	$2ac-cb=ba$
$7\div\overline{2a-c}$	8	$\frac{ba}{2a-b}=c$, the third Term.

If there are four Terms in Musical Proportion, the first hath the same Ratio to the fourth, as the Difference between the first and second hath to the Difference between the third and fourth.

That is, let a, b, c, d, be the four Terms, &c.

Then	1	$a : d :: b-a : d-c$
$1 \because$	2	$db-da=da-ca$
$2+da$	3	$db=2da-ca$
$3\div\overline{2d-c}$	4	$\frac{db}{2d-c}=a$
$3\div d$	5	$b=\frac{2da-ca}{d}$
$3+ca$	6	$db+ca=2da$
$6-db$	7	$ca=2da-db$
$7\div a$	8	$c=\frac{2da-db}{a}$
$7\div\overline{2a-b}$	9	$\frac{ca}{2a-b}=d$

CHAP. VII.

Of Proportion Disjunct, *and how to turn Equations into* Analogies, &c.

PROPORTION Disjunct, or the Rule of Three in Numbers, is already explain'd in *Chap.* 7. *Part* 1. And what hath been there said, is applicable to all Homogeneous Quantities, *viz.* of Lines to Lines, &c.

SECT. 1.

IF four Quantities, (*viz.* either Lines, Superficies, or Solids) be Proportional: the Rectangle comprehended under the Extreams, is equal to the Rectangle comprehended under the two Means. (16 *Euclid* 6.)

For Instance, Suppose, $a.b.c.d.$ to represent the four Homogeneal Quantities in Proportion, *viz.* $a:b::c:d$; then will $ad=bc$. For suppose $b=2a$, then will $d=2c$, and it will be $a:2a::c:2c$. Here the Ratio is 2. But $a\times 2c=2a\times c$. *viz.* $2ca=2ac$. Or suppose $b=3a$, then will $d=3c$, and it will be $a:3a::c:3c$. Here the Ratio is 3. But $a\times 3c=3a\times c$. *viz.* $3ca=3ac$. Or universally putting e for the Ratio of the Proportion, *viz.* making $b=ae$, then will $d=ce$, and it will be $a:ae::c:ce$. But $a\times ce=ae\times c$, viz. $ace=aec$. Consequently, $ad=bc$, which was to be proved.

Whence it follows, that if any three of the four Proportional Quantities be given, the fourth may be easily found; thus,

Let	1	$a:b::c:d$
$1 \because$	2	$ad=bc$ as before.
$2\div d$	3	$ad=\frac{bc}{d}$
$2\div c$	4	$b=\frac{ad}{c}$
$2\div b$	5	$c=\frac{ad}{b}$
$2\div a$	6	$d=\frac{bc}{a}$
$2\div bd$	7	$\frac{a}{b}=\frac{c}{d}$
Or $2\div ac$	8	$\frac{b}{a}=\frac{d}{c}$

Note, In this Manner *Euclid*, in his 5th Book, expresses the Ratio of Proportionals, *viz.* the Ratio of a to b is $\frac{a}{b}$

If four Quantities are Proportionals they will also be Proportionals in Alternation, Inversion, Composition, Division, Conversion, and Mixtly. *Euclid* 5. *Def.* 12, 13, 14, 15, 16.

That

That is if	1	$a : b :: c : d$ be in direct Proportion, as before.
Then	2	$a : c :: b : d$, alternate. For $ad=bc$.
And	3	$b : a :: d : c$, inverted. For $ad=bc$.
Also	4	$a+b : b :: c+d : d$; compounded.
4 ∵	5	$da+bd=bc+bd$, that is, $ad=bc$, as before.
Or	6	$a+c : c :: b+d : d$; alternately compounded.
6 ∵	7	$ad+cd=bc+cd$, that is, $ad=bc$.
Again,	8	$a-b : b :: c-d : d$, divided.
8 ∵	9	$ad-bd=bc-bd$, that is, $ad=bc$.
Or	10	$a-c : c :: b-d : d$, alternately divided.
10 ∵	11	$ad-cd=bc-cd$, that is, $ad=bc$.
And	12	$a : b+a :: c : d+c$, converted.
12 ∵	13	$ad+ac=bc+cd$, that is, $ad=bc$.
Lastly	14	$a+b : a-b :: c+d : c-d$, mixtly.
14 ∵	15	$ac-ad+bc-bd=ac+ad-bc-bd$.
15	16	$2bc=2ad$, that is, $ad=bc$; as at first.

Note, What has been here done about whole Quantities in Simple Proportion, may be easily perform'd in Fractional Quantities, and Surds, *&c.*

For Instance, If $\frac{ab}{c} : \frac{d-c}{f} :: \frac{d+c}{c}$, and if it be required to find the fourth Term, it will be $\frac{dd-cc}{fc}$ the Rectangle of the Means; which being divided by the first Extream $\frac{ab}{c}$ will become $\frac{ab}{c}\Big) \frac{dd-cc}{fc} \Big(\frac{ddc-ccc}{abcf} = \frac{dd-cc}{abf}$ the fourth Term.

Or if $b : \sqrt{bd+bc} :: \sqrt{bd+bc}$: to a fourth Term. Then is, $\sqrt{bd+bc} \times \sqrt{bd+bc} = bd+bc$ the Rectangle of the Means; and $b)\ bd+bc\ (d+c$ the fourth Term. That is, $b : \sqrt{bd+bc} :: \sqrt{bd+bc} : d+c$, &c.

Sect. 2. *Of* Duplicate *and* Triplicate Proportion.

THE Proportions treated of in the last Section, are to be understood when Lines are compared to Lines, and Superficies to Superficies, or Solids to Solids, *viz.* when each is compared to that of its like Kind, which is only called Simple Proportion.

But

But when Lines are compared to Superficies, or Lines are compared to Solids, ſuch Compariſons are diſtinguiſhed from the former, by the Names of Duplicate and Triplicate, (*&c.*) Proportions; ſo that Simple, Duplicate, and Triplicate, *&c.* Proportions are to be underſtood in a different Senſe from Single, Double, Treble, &c. Proportions, which are only as 1, 2, 3, *&c.* to 1; but thoſe of Simple, Duplicate, Triplicate, *&c.* Proportions, are thoſe of $a \,.\, aa \,.\, aaa$. &c. to 1. Or if the ſimple Proportions be that of a to b, whoſe Ratio or Exponent is $\frac{a}{b}$ or $\frac{b}{a}$ according to *Euclid's* Way.

$$\left.\begin{array}{l} \text{Then } \frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb} \text{ is the Exponent of the Duplicate} \\ \text{And } \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a3}{b3} \text{ is the Exponent of the Triplicate} \end{array}\right\} \text{of } \frac{a}{b}$$

Proportions, *&c.*

And if there are three four, or more Quantities in $\because$, as $1 \,.\, a \,.\, aa \,.\, aaa \,.\, a4 \,.\, a5$, &c. (as in the firſt Series, Sect. 2. of the laſt Chapter). Then that of the firſt to the third, fourth, and fifth, *&c.* (*viz.* 1 to $aa \,.\, aaa \,.\, a4 \,.\, a5$) is Duplicate, Triplicate, Quadruplicate, *&c.* of the firſt to the ſecond (*viz.* of 1 to a;) and by Inverſion, that of the third, fourth, fifth, is Duplicate, Triplicate, *&c.* of that of the ſecond to the firſt (a to 1.) *per Def.* 10. *Eucl.* 5. But the Name of theſe Proportions will appear more evident, and be eaſier underſtood when they are applied to Practice, and illuſtrated by Geometrical Figures, further on.

Sect. 3. *How to turn Equations into* ANALOGIES.

FROM the firſt Section of this Chapter, it will be eaſy to conceive how to turn or diſſolve Equations into Analogies or Proportions. For if the Rectangle of two (or more) Quantities, be equal to the Rectangle of two (or more Quantities; then are thoſe four (or more) Quantities Proportional. By the 16 *Eucl.* 6. That is, if $ab = cd$, then is $a : c :: d : b$, or $c : a :: b : d$, &c. From whence there ariſes this general Rule for turning Equations into Analogies.

 RULE.

RULE.

Divide either Side of the given Equation (if it can be done) into two such Parts, or Factors, as being multiplied together, will produce that Side again; and make those two Parts the two Extreams. Then divide the other Side of the Equation if it can be done) in the same Manner as the first was, and let those two Parts or Factors be the two Means.

For Instance, Suppose $ab+ad=bd$. Then $a:b::d:b+d$, or $b:a::b+d:d$, &c. Or taking $a\,d$ from both Sides of the Equation, and it will be $ab=bd-ad$; then $a:d::b-a:b$, or $b:d::b-a:a$, &c.

Again, suppose $aa+2ae=2by+yy$. Here a and $a+2e$ are the two Factors of the first Side in this Equation; for $\overline{a+2e}\times a=aa+2ae$.

Again, y and $2b+y$ are the two Factors of the other Side; therefore, $a:y::2b+y:a+2e$, or $2b+y:a+2e::a:y$, &c.

When one Side of any Equation can be divided into two Factors, as before; and the other Side cannot be so divided, then make the Square Root of that Side either the two Extreams, or the two Means. For Instance, Suppose $bc+bd=da+g$, then $b:\sqrt{da+g}::\sqrt{da+g}:c+d$, or $\sqrt{da+g}:b::c+d:\sqrt{da+g}$, &c.

CHAP. VIII.

Of SUBSTITUTION, *and the Solution of* QUADRATICK EQUATIONS.

Sect. 1. *Of* SUBSTITUTION.

WHEN new Quantities, not concerned in the first Stating of any Question, are put instead of some that are engaged in it, that is called *Substitution*. For Instance, If instead of $\sqrt{bc-dc}$ you put z, or any other Letter; that is, make $z=\sqrt{bc-dc}$ Or suppose $aa+ba-ca+da=dc$, instead of $b-c+d$ put s, or any other Letter not engaged with the Question, viz. $s-b-c+d$, then $aa+sa=dc$. That is, if c be greater than

than $b+d$, it is $aa-sa=dc$; but if $b+d$ be greater than c, then it is $aa+sa=dc$.

And this Way of ſubſtituting or putting of new Quantities inſtead of others, may be found very uſeful upon ſeveral Occaſions; *viz.* in order to make ſome following Operations in the Queſtion more eaſy, and perhaps much ſhorter than they would be without it, as you may obſerve in ſome Queſtions hereafter propoſed in this Tract.

And when thoſe Operations, in which the ſuſtituted Quantities were aſſiſting or uſeful, are performed according as the Nature of the Queſtion required, you may then (if there be Occaſion) bring the original or firſt Quantities into the Equation, in the Place (or Places) of thoſe ſubſtituted Quantities, which is called Reſtitution, as you may ſee further on.

Sect. 2. *The Solution of* Quadratick Equations.

WHEN the Quantity ſought is brought to an Equality with thoſe that are known, and is on one Side of the Equation, in no more than two different Powers whoſe Indices are double one to another, thoſe Equations are called Quadratick Equations Adfected; and do fall under the Conſideration of three Forms or Caſes.

$$\left.\begin{array}{l}\text{Caſe 1. } aa+2ba=dc.\\ \text{Caſe 2. } aa-2ba=dc.\\ \text{Caſe 3. } 2ba-aa=dc.\end{array}\right\} \qquad \text{And} \left\{\begin{array}{l}a^4+2ba^2=dc.\\ a^4-2ba^2=dc.\\ 2ba^2-a^4=dc.\end{array}\right.$$

$$\text{Alſo} \left\{\begin{array}{l}a^6+2ba^3=dc.\\ a^6-2ba^3=dc.\\ 2ba^3-a^6=dc.\end{array}\right\} \qquad \text{And} \left\{\begin{array}{l}a^8+2ba^4=dc.\\ a^8-2ba^4=dc.\\ 2ba^4-a^8=dc.\end{array}\right\} \text{\&c.}$$

When there happens to be more Terms in one of theſe Kind of Equations than two, and the higheſt Power of the unknown Quantity is multiplied into ſome known Co-efficients; you muſt reduce them by *Diviſion*; as in Sect. 4. of Chap. 5, and for the Fractional Quantities that may ariſe by thoſe Diviſions, ſubſtitute another Quantity doubled.

For Inſtance, $baa+caa-ca-da=dc+cb$, then $aa-\dfrac{ca-da}{b+c}=\dfrac{dc+cb}{b+c}$. Make $\dfrac{c-d}{b+c}=2x$, and if you pleaſe,

for $\frac{dc+cb}{b+c}$ put z. Then will $aa-2xa=z$ be the new Equation, equal to the other, being now fitted for a Solution.

Now any of these three Forms of Equations being thus prepared for a Solution, may be reduced to simple Powers by casting off the second or lowest Term of the unknown Quantity; which is done by Substitution; thus, always take half the known Co-efficient, and add it to (Case 1.) or substract it from (Case 2.) its fellow Factor; and for their Sum, or Difference, Substitute another Letter; as in these.

Let	1	$aa+2ba=dc$ Case 1.
Put	2	$a+b=e$
2 ⓖ 2	3	$aa+2ba+bb=ee$
3 − 1	4	$bb=ee-dc$
4 + dc	5	$ee=bb+dc$
5 ɯ 2	6	$e=\sqrt{bb+dc}$
2 and 6	7	$a+b=\sqrt{bb+dc}$, *per Axiom* 5.
7 − b	8	$a=\sqrt{bb+dc}:-b$

Again.

Let	1	$aa-2ba=dc$ Case 2.
Put	2	$a-b=e$
2 ⓖ 2	3	$aa-2ba+bb=ee$
3 − 1	4	$bb=ee-dc$
4 + dc	5	$ee=dc+bb$
5 ɯ 2	6	$e=\sqrt{dc+bb}$
2 and 6	7	$a-b=\sqrt{dc+bb}$
7 + b	8	$a=b+\sqrt{ac+bb}$

In Case 3. From Half the known Co-efficient substract its fellow Factor.

Thus, Let	1	$2ba-aa=dc$
Put	2	$b-a=e$
2 ⓖ 2	3	$bb-2ba+aa=ee$
1 + 3	4	$bb=dc+ee$
4 − dc	5	$ee=bb-dc$

5 ɯ ^{2}	6	$e = \sqrt{bb - dc}$
2 and 6	7	$b - a = \sqrt{bb - dc}$
$7 + a$	8	$b = a + \sqrt{bb - dc}$
$8 - \surd$, &c.	9	$a = b - \sqrt{bb - dc}$

And this Method holds good in thoſe other Equations, wherein the higheſt Powers are a^4, a^6, a^8, &c. As, for Inſtance,

Let	1	$a^6 + 2b^3 = dc$ Caſe 1.
Put	2	$a^3 + b = e$
2 ⊙ ^{2}	3	$a^6 + 2ba^3 + bb = ee$
$3 - 1$	4	$bb = ee - dc$
$4 + cd$	5	$ee = bb + dc$
5 ɯ ^{2}	6	$e = \sqrt{bb + dc}$
2 and 6	7	$a^3 + b = \sqrt{bb + dc}$
$7 - b$	8	$a^3 = \sqrt{bb + dc} : - b$
8 ɯ ^{3}	9	$a = \sqrt[3]{: \sqrt{bb + dc} : - b}$

The ſame may be done with all the reſt, Care being taken to add, or ſubſtract, according as the Caſe requires.

But all Quadratick Equations may be more eaſily reſolved by compleating the Square, which is grounded upon the Conſideration of raiſing a Square from any Binomial, or Reſidual Root. (See Sect. 5. Chap. 1.) *Viz.* if $a + b$ be involved to a Square, it will be $aa + 2ba + bb$; and if $a - b$ be ſo involved, it will be $aa - 2ba + bb$. Whence it is eaſy to obſerve, that $aa + 2ba = dc$ (Caſe 1), and $aa - 2ba = dc$ (Caſe 2), are imperfect Squares, wanting only bb to make them compleat. And therefore it is, that if half the known Co-efficient be involved to the ſecond Power, and the Square be added to both Sides of the Equation, the unknown Side will become a compleat Square.

Thus Let	1	$aa + 2ba = dc$	Here half the Co-efficient $2b$ is b, which being ſquared, is bb.
But	2	$bb = bb$	
$1 + 2$	3	$aa + 2ba + bb = dc + bb$ Caſe 1.	
3 ɯ ^{2}	4	$a + b = \sqrt{dc + bb}$, as before.	

Again

Again.

Let	1	$aa - 2ba = dc$ Case 2.
But	2	$bb = bb$
$1+3$	3	$aa - 2ba + bb = dc + bb$
3ɯ^{2}	4	$a - b = \sqrt{dc + bb}$, &c. as before.

But in Caſe 3. you muſt change the Signs of all the Terms in the Equation,

Thus	1	$2ba - aa = dc$ Caſe 3.
$1 \pm$	2	$aa - 2ba = -dc$
Then	3	$aa - 2ba + bb = bb - dc$, &c.

And this Method of compleating the Square, holds true in thoſe other Equations.

Viz.	1	$aaaa + 2baa = dc$ Caſe 1.
For	2	$bb = bb$, as before.
$1+2$	3	$aaaa + 2baa + bb = dc + bb$
3ɯ^{2}	4	$aa + b = \sqrt{dc + bb}$
$4 - b$	5	$aa = \sqrt{dc + bb} : - b$
5ɯ^{2}	6	$a = \sqrt{: \sqrt{dc + bb} : - b}$, and ſo on for the reſt.

Or let	1	$a^{6} + 2baaa = dc$, as before, Caſe 1.
And	2	$bb = bb$
$1+2$	3	$a^{6} + 2baaa + bb = dc + bb$
1ɯ^{2}	4	$aaa + b = \sqrt{dc + bb}$
$4 - b$	5	$aaa = \sqrt{dc + bb} : - b$
5ɯ^{2}	6	$a = \sqrt[3]{: \sqrt{dc + bb} : - b}$, &c.

COROLLARY.

Hence it is evident, that whatſoever Method is uſed in ſolving theſe (or indeed any other) Equations, the Reſult will ſtill be the ſame, if the Work be true; as you may obſerve from the Operations of this Section: for both theſe Methods here propoſed, give the ſame Theorems in their reſpective Caſes for the Value of (a).

Thus

Thus, when, $aa+2ba=dc$, then

Theorem 1. $a=\sqrt{dc+bb}:-b$

And when $aa-2ba=dc$, then

Theorem 2. $a=b+\sqrt{dc+bb}$

Again, when $2ba-aa=dc$, then

Theorem 3. $a=b-\sqrt{bb-dc}$

The like *Theorems* may be easily raised for the rest.

If the known Co-efficients (of the second or lowest Term) be any single Quantity, as $aa+ba=dc$, &c. then is $\frac{1}{2}b$ its Half, and $\frac{1}{4}bb$ will be the Square of that Half; that is, $\frac{1}{2}b\times\frac{1}{2}b=\frac{1}{4}bb$, and then the Work will stand

Thus	1	$aa+ba=dc$
1 C □	2	$aa+ba+\frac{1}{4}bb=dc+\frac{1}{4}bb$
$2 ㄩ^2$	3	$a+\frac{1}{2}b=\sqrt{dc+\frac{1}{4}bb}$
$3-\frac{1}{2}b$	4	$a=\sqrt{dc+\frac{1}{4}bb}:-\frac{1}{2}b$, and so for the rest.

Note, C □ placed in the Margin against the second Step, signifies that the imperfect Square $aa+ba$ in the first Step, is there compleated, *viz.* in the second Step.

Now by the Help of these *Theorems*, it will be easy to calculate or find the Value of the unknown Quantity (a) in Numbers.

EXAMPLE 1.

Suppose $aa+2ba=z$. Let $b=16$, and $z=4644$.

Then $a=\sqrt{z+bb}:-b$ per *Theorem* 1.

But $z+bb=4644+256=4900$, and $\sqrt{4900}=70$

Consequently $a=70-16$, viz. $a=54$.

But every Adfected Equation, hath as many Roots (or rather Values of the unknown Quantity) either real or imaginary, as are the Dimensions (*viz.* the Index) of its highest Power; and therefore the Quantity a, in this Equation hath another Value either Affirmative or Negative; which may be thus found.

The given Equation is $aa+32a=4644$, and its Root $a=54$.

Let these two Equations be made equal or equated to 0, *viz.* to Nothing.

Thus,

Thus, $aa+32a-4644=0$, and $a-54=0$.

Then divide the given Equation by its firſt Root, and the Quotient will ſhew the ſecond Value of a.

$$
\begin{array}{l}
\text{Thus, } a-54=0)\ aa+32a-4644=0\ (a+86=0 \\
\qquad\qquad\qquad\ \ aa-54a \\
\qquad\qquad\qquad\ \ \overline{\quad +86a-4644} \\
\qquad\qquad\qquad\qquad\ 86a-4644 \\
\qquad\qquad\qquad\ \ \overline{\qquad\quad (0)\qquad}
\end{array}
$$

Hence the ſecond Value of a is $=-86$, or $86=-a$ which ſeems impoſſible, *viz.* that an Affirmative Quantity ſhould be equal to a Negative Quantity; yet even by this ſecond Value of a, and the ſame Co-efficient, the true (or firſt) Equation may be formed.

Thus, Let	1	$a=-86$
1 ⓖ ^{2}	2	$aa=+7396$, viz. $-86\times-86=+7396$
1×32	3	$32a=-2752$
$2+3$	4	$aa+32a=4644$, as at firſt.

EXAMPLE 2.

Suppoſe	1	$aa-7a=948,75$, then *per Theorem* 2.
1 C □	2	$aa-7a+\frac{4}{4}^{2}=948,75+\frac{4}{4}^{2}=961$.
$2 ɯ^2$	3	$a-{}^{7}$ (or 3,5)$=\sqrt{961}=31$
$3+\overline{3,5}$	4	$a=31+3,5=34,5$

Again, for the ſecond Value of a, let $aa-7a-948,75=0$, and $a-34,5=0$. Then

$$a-34,5=0)\ aa-7a-948,75=0\ (a+27,5=0.$$

Conſequently this ſecond Value is $a=-27,5$ which will form the original Equation, $aa-7a=948,75$ if it be ordered as the laſt was.

EXAMPLE 3.

Suppoſe $36a-aa=243$, then *per Theorem* 3. $a=18-\sqrt{324-243}$, *viz.* half 36 ſquared is 324, &c. that is, $a=18-\sqrt{81}$; but $\sqrt{81}=9$, therefore $a=18-9=9$. Now this third Form is called an ambiguous Equation, becauſe it hath two Affirmative Values of the unknown Quantity *(a)*, both which may be found without ſuch Diviſion as was uſed before.

before. For in this Case, $a=18+\sqrt{81}$, viz. $a=18+9=27$, or, $a=18-9=9$, as before. And both these Values of a are equally true, as to forming the given Equation; *viz.* $36a-aa=243$. For if $a=9$, then $aa=81$, and $36a=324$; but $324-81=243$, therefore $a=9$.

Again, if $a=27$, then will $aa=729$, and $36a=972$: But $972-729=243$, consequently it may be, $a=27$. Now either of these Values of a may be found by Division, as those were in the other two Cases, one of them being first found by the Theorem. Thus, let $36a-aa-243=0$, and $9-a=0$, then $9-a=0$) $36a-aa-243=0$ ($a-27=0$

$$\begin{array}{l} 9a-aa \\ \hline 27a-0-243 \\ 27 \quad\quad 243 \\ \hline (0) \quad\quad (0) \end{array}$$

Hence, if $a-27=0$, then $a=27$, as before.

Notwithstanding all Quadratick Equations of this third Form have two Affirmative Roots (as in this) yet but one of those Roots will give a true Answer to the Question, and that is to be chosen according to the Nature and Limits of the Question, as shall be shewed further on.

SCHOLIUM.

From the Work of the three last Examples, it may be observed; that the Sum of both the Roots will always be equal to the Co-efficient of their respective Equations, with a contrary Sign.

Thus, In *Example* 1. $aa+32a=4644$

$$\left.\begin{array}{l} \text{Here } a=\ \ 54 \\ \text{Add } a=-86 \end{array}\right\} \text{Add}$$
$$2a=-32$$

In *Example* 2. $aa-7a=948,75$

$$\left.\begin{array}{l} \text{Here } a=\ \ 34,5 \\ \text{And } a=-27.5 \end{array}\right\} \text{Add}$$
$$2a=+7$$

In the last *Example* $36a-aa=243$

Which was changed into $aa-36a=-243$

$$\left.\begin{array}{l} \text{Here } a=\ 9 \\ \text{And } a=27 \end{array}\right\} \text{Add}$$
$$2a=36$$

Hence

Hence it is evident, that if either the Roots be found, the other may be easily had without Divisions.

If the Contents of this Section be well understood, it will be easy to give a Numerical Solution to any Quadratick Equation, that happens to arise in resolving of Questions, &c. And as for giving a Geometrical Construction of them, I think it not proper in this Place; because I here suppose the Learner wholly ignorant of the first Principles of Geometry, therefore I shall refer that Work to the next Part.

CHAP. IX.

Of ANALYSIS, *or the Method of resolving* PROBLEMS *exemplified by Variety of Numerical* QUESTIONS.

N.B. HERE *I advise the Learner to make use always of the same Letters, to represent the same* Data *in all Questions.*

Viz. $\left\{\begin{array}{l}\text{If } a \text{ represent any Number}\\ \text{And } e \text{ represent a less Number}\end{array}\right\}$ or other Quantity,

Then let $\left\{\begin{array}{ll} a+e=s & \text{their Sum.}\\ a-e=d & \text{their Difference.}\\ ae=p & \text{their Product.}\\ \dfrac{a}{e}=q & \text{their Quotient.}\\ aa+ee=z & \text{the Sum of their Squares.}\\ aa-ee=x & \text{the Difference of their Squares.}\end{array}\right.$

Any two of these six (s, d, p, q, z, x) being given, thence to find the rest; which admits, of fifteen Variations, or Questions.

Question 1. Suppose s and d were given, and it were required by them to find $a.\ e.\ p.\ q.\ z.$ and x.

Let	1	$a+e=s$
	2	$a-e=d$	and suppose $\left\{\begin{array}{l} s=240\\ d=192\end{array}\right\}$ Then
$1+2$	3	$2a=s+d=432$
$3\div 2$	4	$a=\dfrac{s+d}{2}=216$, here a is found.
$1-2$	5	$2e=s-d=48$.

5 ÷ 2

$5 \div \overline{2}$	6	$e = \frac{s-dd}{2} = 24$, here e is found.
4×6	7	$ae = \frac{ss-dd}{4} = p = 5184$, here p is found.
$4 \div 6$	8	$\frac{a}{e} = \frac{s+d}{s-d} = q = 9$, here q is found.
$4 \odot^{2}$	9	$aa = \frac{ss+2sd+dd}{4} = 46656$
$6 \odot^{2}$	10	$ee = \frac{ss-2sd+dd}{4} = 576$
$9+10$	11	$aa+ee = \frac{ss+dd}{2} = z = 47232$, z found.
$9-10$	12	$aa-ee = sd = x = 46080$, x found.

Question 2. Let s and p be given to find the rest.

That is	1	$a+e=s. = 240$ Quære $a.e.d.q.z.x.$
	2	$ae=p=5184$
$1 \odot^{2}$	3	$aa+2ae+ee=ss=$ 57600
2×4	4	$4ae=4p$ 20736
$3-4$	5	$aa-2ae+ee=ss-4p=36864$
$5\,\omega^{2}$	6	$a-e=\sqrt{ss-4p}=d=192$
$1+6$	7	$2a=s+\sqrt{ss-4p}$
$7 \div \overline{2}$	8	$a=\frac{s+\sqrt{ss-4p}}{2}$, hence $a=216$
$1-6$	9	$2e=s-\sqrt{ss-4p}$
$9 \div \overline{2}$	10	$e=\frac{s-\sqrt{ss-4p}}{2}$ hence $e=24$
$8 \div 10$	11	$\frac{a}{e}=\frac{s+\sqrt{ss+4p}}{s-\sqrt{ss-4p}}=q=9$
$8 \odot^{2}$	12	$aa=\frac{ss+s\sqrt{ss-4p}}{2}:-p$
$10 \odot^{2}$	13	$ee=\frac{ss-s\sqrt{ss-4p}}{2}:-p$
$12+13$	14	$aa+ee=ss-2p=z=47232$
$12-13$	15	$aa-ee=s\sqrt{ss-4p}=x=46080$

Question 3. Suppose s and q, are given to find the rest.

Viz.	1	$a+e=s. =240$ } Quære $a.e.d.p.z.x.$
	2	$\frac{a}{e}=q=9$
$2\times e$	3	$a=qe$
$1-3$	4	$e=s-qe$
$4+qe$	5	$qe+e=s$
$5\div\overline{q+1}$	6	$e=\frac{s}{q+1}$, for $\overline{q+1}\times e=qe+e$
$1-6$	7	$a=s-\frac{s}{q+1}=\frac{qs}{q+1}$
6×7	8	$ae=\frac{qss}{qq+2q+1}=p$
$7-6$	9	$a-e=\frac{qs-s}{q+1}=d$
$7\,\circledcirc^{2}$	10	$aa=\frac{qqss}{qq+2q+1}$
$6\,\circledcirc^{2}$	11	$ee=\frac{ss}{qq+2q+1}$
$10+11$	12	$aa+ee=\frac{qqss+ss}{qq+2q+1}=z$
$10-11$	13	$aa-ee=\frac{qqss-ss}{qq+2q+1}=x.$

Question 4. Let s and z be given, to find the rest.

Viz.	1	$a+e=s. =240$ } Quære $a.e\,d.p.q.x.$
	2	$aa+ee=z. =47232$
$1\,\circledcirc^{2}$	3	$aa+2ae+ee=ss$
$3-2$	4	$2ae=ss-z$
$2-4$	5	$aa-2ae+ee=2z-ss$
$5\,\omega^{2}$	6	$a-e=\sqrt{2z-ss}=d$

$1+6$	7	$2a=s+\sqrt{2z-ss}$
$7\div\overline{2}$	8	$a=\dfrac{s+\sqrt{2z-ss}}{2}$
$1-6$	9	$2e=s-\sqrt{2z-ss}$
$9\div\overline{2}$	10	$e=\dfrac{s-\sqrt{2z-ss}}{2}$

The reſt are found juſt as in the 2d *Queſtion*; the 8 and 10 Steps here being the very ſame with the 8 and 10 Steps there.

Queſtion 5. When s and x are given, to find the reſt.

Viz.	1	$a+e=s=240$
	2	$aa-ee=x=46080$

Quære $a.e.d.p.q.z.$

$2\div1$	3	$a-e=\dfrac{x}{s}=d$, viz. $a+e)\ aa-ee\ (a-e$
$1+3$	4	$2a=s+\dfrac{x}{s}=\dfrac{ss+x}{s}$
$4\div\overline{2}$	5	$a=\dfrac{ss+x}{2s}$
$1-3$	6	$2e=s-\dfrac{x}{s}=\dfrac{ss-x}{s}$
$6\div\overline{2}$	7	$e=\dfrac{ss-x}{2s}$
5×7	8	$ae=\dfrac{ssss-xx}{4ss}=p$
$5\div7$	9	$\dfrac{a}{e}=\dfrac{ss+x}{ss-x}=q$
$5\odot^{2}$	10	$aa=\dfrac{s^4+2ssx+xx}{4ss}$
$7\odot^{2}$	11	$ee=\dfrac{s^4-2ssx+xx}{4ss}$
$10+11$	12	$aa+ee=\dfrac{s^4+xx}{2ss}=z$

Queſtion

Question 6. Suppose d and p are given, to find the rest.

Viz.	1	$a-e=d=192$
	2	$ae=p=5184$ } Quære $a.e.s.q.z.x.$
$1 \circledcirc^2$	3	$aa-2ae+ee=dd$
2×4	4	$4ae=4p$
$3+4$	5	$aa+2ae+ee=dd+4p$
$5\,ɯ^2$	6	$a+e=\sqrt{dd+4p}=s$
$6+1$	7	$2a=d+\sqrt{dd+4p}$
$7\div\overline{2}$	8	$a=\dfrac{d+\sqrt{dd+4p}}{2}$
$6-1$	9	$2e=\sqrt{dd+4p}-d$
$9\div\overline{2}$	10	$e=\dfrac{\sqrt{dd+4p}-d}{2}$
$8\div10$	11	$\dfrac{a}{e}=\dfrac{d+\sqrt{dd+4p}}{\sqrt{dd+4p}-d}=q$
$8 \circledcirc^2$	12	$aa=\dfrac{dd+2p+d\sqrt{dd+4p}}{2}$
$10 \circledcirc^2$	13	$ee=\dfrac{dd+2p-d\sqrt{dd+4p}}{2}$
$12+13$	14	$aa+ee=dd+2p=z$
$12-13$	15	$aa-ee=d\sqrt{dd+4p}=x$

Question 7. Let d and q be given, to find the rest.

Viz.	1	$a-e=d=192$
	2	$\dfrac{a}{e}=q=9$ } Quære $a.e.s.p.z.x.$
$2\times e$	3	$a=qe$
$1+e$	4	$a=d+e$
3 and 4	5	$qe=d+e$
$5-e$	6	$qe-e=d$
$6\div\overline{q-1}$	7	$e=\dfrac{d}{q-1}$, for $\overline{q-1}\times e=qe-e$
$1+7$	8	$a=d+\dfrac{d}{q-1}=\dfrac{qd}{q-1}$
$7+8$	9	$a+e=\dfrac{qd+d}{q-1}=s$

7×8

7×8	10	$ae = \frac{qdd}{qq-2q+1} = p$
$8 \odot^{2}$	11	$aa = \frac{qqdd}{qq-2q+1}$
$7 \odot^{2}$	12	$ee = \frac{dd}{qq-2q+1}$
$11+12$	13	$aa + ee = \frac{qqdd + dd}{qq-2q+1} = z$
$11-12$	14	$aa - ee = \frac{qqdd - dd}{qq-2q+1} = x$

Question 8. Suppose d and z given, to find the rest.

Viz.	1	$a - e = d = 192$
	2	$aa + ee = z = 47232$ Quære $a . e . s . p . q . x.$
$1 \odot^{2}$	3	$aa - 2ae + ee = dd$
$2-3$	4	$2ae = z - dd$
$2+4$	5	$aa + 2ae + ee = 2z - dd$
$5 ɯ^{2}$	6	$a + e = \sqrt{2z - dd} = s$
$1+6$	7	$2a = d + \sqrt{2z - dd}$
$7 \div 2$	8	$a = \frac{d + \sqrt{2z - dd}}{2}$
$6-1$	9	$2e = \sqrt{2z - dd} - d$
$9 \div 2$	10	$e = \frac{\sqrt{2z - dd} - d}{2}$
8×10	11	$ae = \frac{z - dd}{2} = p$
$8 \odot^{2}$	12	$aa = \frac{z + d\sqrt{2z - dd}}{2}$
$10 \odot^{2}$	13	$ee = \frac{z - d\sqrt{2z - dd}}{2}$
$12-13$	14	$aa - ee = d\sqrt{2z - dd} = x$
$8 \div 10$	15	$\frac{a}{e} = \frac{d + \sqrt{2z - dd}}{\sqrt{2z - dd} - d} = q$

Question

Question 9. Let d and x be given to find the rest.

Viz.	1	$a - e = d = 192$
	2	$aa - ee = x = 46080$ } Quære $a \,.\, e \,.\, s \,.\, p \,.\, q \,.\, z$.
$2 \div 1$	3	$a + e = \dfrac{x}{d} = s$, viz. $a - e)\; aa - ee\; (a + e$
$1 + 3$	4	$2a = \dfrac{dd + x}{d}$
$4 \div 2$	5	$a = \dfrac{dd + x}{2d}$
$3 - 5$	6	$e = \dfrac{x - dd}{2d}$
5×6	7	$ae = \dfrac{xx - d^4}{4dd} = p$
$5 \div 6$	8	$\dfrac{a}{e} = \dfrac{dd + x}{x - dd} = q$
$5 \odot^{2}$	9	$aa = \dfrac{d^4 + 2ddx + xx}{4dd}$
$6 \odot^{2}$	10	$ee = \dfrac{xx - 2ddx + d^4}{4dd}$
$9 + 10$	11	$aa + ee = \dfrac{d^4 + xx}{2dd} = z$

Question 10. Let p and q be given, to find the rest.

Viz.	1	$ae = p = 5184$
	2	$\dfrac{a}{e} = q = 9$ } Quære $a \,.\, e \,.\, s \,.\, d \,.\, z \,.\, x$.
1×2	3	$aa = qp$, for $\dfrac{ae}{1} \times \dfrac{a}{e} = \dfrac{aae}{e} = aa$
$3 \omega^{2}$	4	$a = \sqrt{pq}$
$1 \div 2$	5	$ee = \dfrac{p}{q}$, for $\dfrac{a}{e}) \dfrac{ae}{1} (\dfrac{aee}{a} = ee$
$5 \omega^{2}$	6	$e = \sqrt{\dfrac{p}{q}}$
$4 + 6$	7	$a + e = \sqrt{pq} + \sqrt{\dfrac{p}{q}} = s$

4—6

4—6	8	$a-e=\sqrt{qp}-\sqrt{\frac{p}{p}}=d$
3+5	9	$aa+ee=qp+\frac{p}{q}=x$
3—5	10	$aa-ee=pq-\frac{p}{q}=x$

Question 11. Let p and z be given; to find the rest.

Viz.	1	$ae=p=5184$	Quære $a.e.$&c.
	2	$aa+ee=z=47232$	
1×2	3	$2ae=2p$	
2+3	4	$aa+2ae+ee=z+2p$	
4ɯ²	5	$a+e=\sqrt{z+2p}=s$	
2—3	6	$aa-2ae+ee=z-2p$	
6ɯ²	7	$a-e=\sqrt{z-2p}=d$	
5+7	8	$2a=\sqrt{z+2p}+\sqrt{z-2p}$	
÷2	9	$a=\frac{\sqrt{z+2p}+\sqrt{z-2p}}{2}$	
5—7	10	$2e=\sqrt{z+2p}+\sqrt{z-2p}$	
10÷2	11	$e=\frac{\sqrt{z+2p}-\sqrt{z-3p}}{2}$	
9÷11	12	$\frac{a}{e}=\frac{\sqrt{z+2p}+\sqrt{z-2p}}{\sqrt{z-2p}-\sqrt{z-2p}}=q$	
9⊙²	13	$aa=\frac{z+\sqrt{zz-4pp}}{2}$	
11⊙²	14	$ee=\frac{z-\sqrt{zz-4pp}}{2}$	
$aa-ee$	15	$aa-ee=\sqrt{zz-4pp}=x$	

Question 12. Let p and x be given, to find the rest.

Viz.	1	$ae=p=5184$	Quære $a.e.$&c.
	2	$aa-ee=x=46080$	
1⊙²	3	$aaee=pp$	

2 ⊚^{2}	4	$aaaa - 2aaee + eeee = xx$
3 × 4	5	$4aaee = 4pp$
4 + 5	6	$aaaa + 2aaee + eeee = xx + 4pp$
6 ωω^2	7	$aa + ee = \sqrt{xx + 4pp} = z$
2 + 7	8	$2aa = x + \sqrt{xx + 4pp}$
8 ÷ $\overline{2}$	9	$aa = \dfrac{x + \sqrt{xx + pp}}{2}$
9 ωω^2	10	$a = \sqrt{\dfrac{x + \sqrt{xx + 4pp}}{2}}$
7 − 2	11	$2ee = \sqrt{xx + 4pp} - x$
11 ÷ $\overline{2}$	12	$ee = \dfrac{\sqrt{xx + 4pp} - x}{2}$
12 ωω^2	13	$e = \sqrt{:\dfrac{\sqrt{xx + 4pp} - x}{2}}$
10 = 13	14	$a + e = \sqrt{\dfrac{x + \sqrt{xx + 4pp}}{2}} + \sqrt{\dfrac{\sqrt{xx + 4pp} - x}{2}} = s$
10 + 13	15	$a - e = \sqrt{\dfrac{x + \sqrt{xx + 4pp}}{2}} - \sqrt{\dfrac{\sqrt{xx + 4pp} - x}{2}} = d$
9 + 12	16	$aa + ee = \sqrt{xx + 4pp} = z$

Question 13. Having q and z given, to find the rest.

Viz.	1	$\dfrac{a}{e} = b = q$ } Quære a . e. &c.
	2	$aa + ee = z = 47232$
1 × e		$a = qe$
2 ⊚^{2}	3	$aa = qqee$
2 − 4	4	$ee = z - qqee$
4 + $qqee$	5	$qqee + ee = z$
6 ÷ $\overline{qq+1}$	6, 7	$ee = \dfrac{z}{qq + 1}$ for $\overline{qq+1} \times ee = qqee + ee$

2—7

2—7	8	$aa=z-\frac{z}{qq+1}=\frac{qqz}{qq+1}$
$8\omega^2$	9	$a=\sqrt{\frac{qqz}{qq+1}}$
$7\omega^2$	10	$e=\sqrt{\frac{z}{qq+1}}$
9+10	11	$a+e=\sqrt{\frac{qqz}{qq+1}}+\sqrt{\frac{z}{qq+1}}=s$
9—10	12	$a-e=\sqrt{\frac{qqz}{qq+1}}-\sqrt{\frac{z}{qq+1}}=d$
9×10	13	$ae=\sqrt{\frac{qqzz}{q^4+2qq+1}}=p$
8—7	14	$aa-ee=\frac{qqz-z}{qq+1}=x$

Queſtion 14. When q and x are given, to find the reſt.

Viz.	1	$\frac{a}{e}=q=9$	Quære a . e . &c.
	2	$aa-ee=x=46080$	
1×e	3	$a=qe$	
3 ω^2	4	$aa=qqee$	
2+ee	5	$aa=x+ee$	
4 and 5	6	$qqee=x+ee$	
6—ee	7	$qqee-ee=x$	
$7\div\overline{qq-1}$	8	$ee=\frac{x}{qq-1}$	
2+8	9	$aa=x+\frac{x}{qq-1}=\frac{qqx}{qq-1}$	
$9\omega^2$	10	$a=\sqrt{\frac{qqx}{qq-1}}$	
$8\omega^2$	11	$e=\sqrt{\frac{x}{qq-1}}$	
10+11	12	$a+e=\sqrt{\frac{x}{qq-1}}+\sqrt{\frac{x}{qq-1}}=s$	

10—11	13	$a-e=\sqrt{\frac{qqx}{qq-1}}-\sqrt{\frac{x}{qq-1}}=d$
10×11	14	$ae=\sqrt{\frac{qqxx}{qqqq-2qq+1}}=p$
8+9	15	$aa+ee=\frac{qqx-x}{qq-1}=z$

Question 15. When z and x are given, to find the reſt.

Viz.	1	$aa+ee=z=47232$ Quære a . e . &c.
	2	$aa-ee=x=46080$
1+2	3	$2aa=z+x$
$3 \div \overline{2}$	4	$aa=\frac{z+x}{2}$
1—2	5	$2ee=z-x$
$5 \div \overline{2}$	6	$ee=\frac{z-x}{2}$
4 ɯ²	7	$a=\sqrt{\frac{z+x}{2}}$
6 ɯ²	8	$e=\sqrt{\frac{z-x}{2}}$
7+8	9	$a+e=\sqrt{\frac{z+x}{2}}+\sqrt{\frac{z-x}{2}}=s$
7—8	10	$a-e=\sqrt{\frac{z+x}{2}}-\sqrt{\frac{z-x}{2}}=d$
7×8	11	$ae=\sqrt{\frac{zz-xx}{4}}=p$
7÷8	12	$\frac{a}{e}=\frac{\sqrt{z+x}}{\sqrt{z-x}}=q$

Theſe fifteen Queſtions are propoſed in Dr. *Pell's Algebra*; but he purſues only the firſt Queſtion throughout, and breaks off in the other fourteen, after the Values of what I call a and e are found. But I have proceeded in every one of them, to find the

the Value of all the unknown Quantities, becauſe they afford ſuch Variety, as being well obſerved by a Learner, will be found very uſeful in the Solution of moſt Queſtions.

Note, I have choſe to uſe the ſame Numbers for the reſpective Value of each Quantity throughout all the Queſtions, becauſe they will be more ſatisfactory in proving the Work than various Numbers would have been. Not but that any Numbers may be taken at Pleaſure, provided that the Number repreſented by *a*, be greater than that by *e*, &c. I have omitted the Numerical Calculations purely for the Learner to practiſe on.

Queſtion 16. There are two Numbers, the Sum of their Squares is 2368; and the greater of them is in Proportion to the leſs, as 6 to 1. What are theſe Numbers?

Let $a =$ the greater Number, $e =$ the leſſer, and $z = 2368$.

Then	1	$aa + ee = z$ } by the Queſtion.
And	2	$a : e :: 6 : 1$ }
$2 \therefore$	3	$1a = 6e$
$3\,\mathcal{G}^2$	4	$aa = 36ee$
$1 - 4$	5	$ee = z - 36ee$
$5 + 36ee$	6	$37ee = z$
$6 \div 37$	7	$ee = \frac{z}{37} = 64$
$7\,\omega^2$	8	$e = \sqrt{\frac{z}{37}} = 8$
8×6	9	$6e = 6\sqrt{\frac{z}{37}} = 48$
3 and 9	10	$a = 48$

Proof.

If $a =$	48
and $e =$	8
$aa =$	2304
$ee =$	64
$aa + ee =$	2362
and $48 : 8 :: 6 : 1$	

Queſtion 17. There are three Numbers in continued Proportion, the Sum of the Extreams is 156, and the Mean is 72; What are the two Extreams?

That is, Suppoſe $a . m . e$ in $\div\!\div$, and $m = 72$.

Then	1	$a + e = 156 = s$ } by the Queſtion.
	2	$a : m :: m : e$ } Quære $a . e .$ &c.
$2 \therefore$	3	$aa = mm$
$1\,\mathcal{G}^2$	4	$aa + 2ae + ee = ss$
3×4	5	$4ae = 4mm$

$4 - 5$	6	$aa - 2ae + ee = ss - 4mm$
$6\, \omega\omega^2$	7	$a - e = \sqrt{ss - 4mm}$
$1 + 7$	8	$2a = s + \sqrt{ss - 4mm}$
$8 \div 2$	9	$a = \dfrac{s + \sqrt{ss - 4mm}}{2} = 108$ } Or { $a = 48$
$1 - 9$	10	$e = \dfrac{s - \sqrt{ss - 4mm}}{2} = 48$ } { $e = 108$

Question 18. There are three Numbers in ∺, their Sum is 74, and the Sum of their Squares is 1924; What are those Numbers?

That is, a, e, y, are in ∺

Then	1	$a + e + y = s = 74$ } Quære a, e, y.
	2	$aa + ee + yy = z = 1924$
	3	$a : e :: e : y$
3 ∵	4	$ay = ee$
$1 - e$	5	$a + y = s - e$
$2 - ee$	6	$aa + yy = z - ee$
4×2	7	$2ay = 2ee$
$6 + 7$	8	$aa + 2ay + yy = z + ee$
$5 \odot^2$	9	$aa + 2ay + yy = ss - 2se + ee$
8 and 9	10	$z + ee = ss - 2se + ee$
$10 - ee$	11	$2se = ss - z$
$11 \div 2s$	12	$e = \dfrac{ss - z}{2s} = 24$
$5 \odot^2$	13	$a + y = s - e = 50$
$13 \odot^2$	14	$aa + 2ay + yy = 2500$
4×4	15	$4ay = 4ee = 2304$
$14 - 15$	16	$aa - 2ay + yy = 196$
$16\, \omega\omega^2$	17	$a - y = \sqrt{196} = 14$
$13 + 17$	18	$2a = 50 + 14 = 64$
$18 \div 2$	19	$a = 32$ } Or { $a = 18$
$13 - 19$	20	$y = 50 - 32 = 18$ } { $y = 32$

Note, in all Questions about continual Proportionals, (either Arithmetical or Geometrical) where three Terms are sought, the Mean is the easiest found first (as above) and if all the Terms be Affirmative, then it is equal whether the first or last Term be the greatest.

Question

Question 19. There are three Numbers in $\div\!\div$ their Sum is 76; and if the Sum of the Extreams be multiplied into the Mean, that Product will be 1248; What are those Numbers?

Viz.	1	$a:e::e:y$	by the Question.
	2	$a+e+y=s=76$	
	3	$ae+ye+p=1248$	
1 ∵	4	$ay=ee$	
$1\times e$	5	$ae+ee+ye=se$	
$5-3$	6	$ee=se-p$	
$6-se$	7	$ee-se=-p$	
7 C □	8	$ee-se+\frac{1}{4}ss=\frac{1}{4}ss-p$	
8 ωω^2	9	$e-\frac{1}{2}s=\sqrt{\frac{1}{4}ss-p}$	
$6+\frac{1}{2}s$	10	$e=\frac{1}{2}s+\sqrt{\frac{1}{4}ss-p}=\left\{\begin{matrix}52\\24.\end{matrix}\right.$	per Theorem 3. Chap. 8.
$2-10$	11	$a+y=52$	
4×4	12	$4ay=4ee=2304$	
11 ⓖ^{2}	13	$aa+2ay+yy=2704$	
$13-12$	14	$aa-2ay+yy=400$	
14 ωω^2	15	$a-y=\sqrt{400}=20$	
$11+15$	16	$2a=52+20=72$	
$16\div 2$	17	$a=36$	Or $a=16$
$11-17$	18	$y=52-36=16$	and $y=36$

N. B. If you take $e=\frac{1}{2}s+\sqrt{\frac{1}{4}ss-p}=52$ (at the 10th Step) then it will be $76-52=24=a+y$, which is impossible, *viz.* that the Mean should be greater than the Sum of the two Extreams. Therefore it must be $e=\frac{1}{2}s-\sqrt{\frac{1}{4}ss-p}=24$. (See page 201.)

Question 20. There are three Numbers in Arithmetical Progression, the first being added to twice the second, and three times the third, their Sum will be 62; and the Sum of all their Squares is 275; What are those Numbers?

Suppose	1	a, e, y in Arithmetical Progression	
And	2	$a+2e+3y=62$	by the Question.
	3	$aa+ee+yy=275$	
Then	4	$a+y=2e$, per Sect. Chap. 6.	
$2-4$	5	$2e+2y=62-2e$	
$5\div 2$	6	$e+y=31-e$	
$6-e$	7	$y=31-2e$	
$4-7$	8	$a=4e-31$	

8 ⓖ^{2}

$8 \circledcirc^2$	9	$aa = 16ee - 248e + 961$
$7 \circledcirc^2$	10	$yy = 961 - 124e + 4ee$
$9 + 10$	11	$aa + yy = 20ee - 372e + 1922$
$3 - 11$	12	$ee = 372e - 20ee - 1647$
$12 + 20ee$	13	$21ee = 372e - 1647$
$13 - 372e$	14	$21ee - 372e = -1647$
$14 \div 21$	15	$ee - \frac{124}{7}e = -\frac{549}{7}$
$15\ C\ \square$	16	$ee - \frac{124}{7}e + \frac{3844}{49} = \frac{3844}{49} - \frac{549}{7} = \frac{1}{49}$
$16 uu$	17	$e - \frac{62}{7} = \sqrt{\frac{1}{49}} = \frac{1}{7}$, the Mean.
$17 + \frac{62}{7}$	18	$e = \frac{62}{7} + \frac{1}{7} = 9$, or $8\frac{5}{7}$
18×4	19	$4e = 36$, or $34\frac{6}{7}$
8 and 19	20	$a = 36 - 31 = 5$, or $34\frac{6}{7} - 31 = 3\frac{6}{7}$
18×2	21	$2e = 18$, or $17\frac{3}{7}$
7 and 21	22	$y = 31 - 18 = 13$, or $31 - 17\frac{3}{7} = 14\frac{4}{7}$

Question 21. There are three Numbers in Arithmetical Progression; the Square of the first Term being added to the Product of the other two is 576; the Square of the Mean being added to the Product of the two Extreams, makes 612; and the Square of the last Term being added to the Product of the first into the second, is 792: What are those Numbers?

Suppose	1	a, e, y in Arith. Progref. as before.
Then	2	$aa + ye = 576$ by the Question.
	3	$ee + ya = 612$ by the Question.
	4	$yy + ae = 792$ by the Question.
$1 \because$	5	$a + y = 2e$, per Sect. 1. Chap. 6.
$5 \times e$	6	$ae + ye = 2ee$
$2 + 4$	7	$aa + ye + yy + ae = 1368$
$7 - 6$	8	$aa + yy = 1368 - 2ee$
$3 - ee$	9	$ya = 612 - ee$
9×2	10	$2ya = 1224 - 2ee$
$8 + 10$	11	$aa + 2ya + yy = 2592 - 4ee$
$5 \circledcirc^2$	12	$aa + 2ya + yy = 4ee$
11 and 12	13	$4ee = 2592 - 4ee$
$13 + 4ee$	14	$8ee = 2592$
$14 \div 8$	15	$ee = 324$
$15 m^2$	16	$e = \sqrt{324} = 18$, the Mean
8,	17	$aa + yy = 1368 - 2ee = 720$
10,	18	$2ya = 1224 - 2ee = 576$
$17 - 18$,	19	$aa - 2ya + yy = 720 - 576 = 144$

Iuu^2

1ω^2	20	$a - y = \sqrt{144} = 12$
$5 + 20$	21	$2a = 2e + 12 = 48$
$21 \div 2$	22	$a = 24$
$5 - 22$	23	$y = 2e - 24 = 12$

Or $\begin{cases} a = 12 \\ y = 24 \end{cases}$

Queſtion 22. It is required to find two ſuch Numbers, that the Sum of their Squares may be $8226\frac{1}{2}$; and their Product being added to the Square of the leſſer; may be $6921\frac{1}{2}$.

Viz.	1	$aa + ee = 8226\frac{1}{2}$ } Quære a and e
	2	$ae + ee = 6921\frac{1}{2}$
$1 - 2$	3	$aa - ae = 1305$
$3 +$	4	$ae = aa - 1305$
$4 \div \overline{a}$	5	$e = \dfrac{aa - 1305}{a}$
5Ⓖ^2	6	$ee = \dfrac{a^4 - 2610aa + 1703025}{aa}$
$1 - aa$	7	$ee = 8226{,}5 - aa$
6 and 7	8	$\dfrac{a^4 - 2610aa + 1703025}{aa} = 8226{,}5 - aa$
$8 \times aa$	9	$a^4 - 2610aa + 1703025 = 8226{,}5aa - a^4$
$9 + a^4$	10	$2a^4 - 2610aa + 1703025 = 8226{,}5aa$
$10 +$	11	$2a^4 - 10836{,}5aa = -1703025$
$11 \div \overline{2}$	12	$a^4 - 5418{,}25aa = -851512{,}5$
$12\ C\ \square$	13	$a^4 - 5418{,}25aa + 7339358{,}265625 = 6487845{,}765$
13ω^2	14	$aa - 2709{,}125 = \sqrt{6487845{,}765625} - 2547{,}125$
$14 + 27$ &c.	15	$aa = 2709{,}125 + 2547{,}125$
Suppoſe	16	$aa = 2709{,}125 + 2547{,}125 = 5256{,}25$
Then	17	$a = \sqrt{5256{,}25} = 72{,}5$
And 5,	18	$e = \dfrac{aa + 1305}{a} = \dfrac{5256{,}25 - 1305 = 54.5}{72{,}5}$
Or let,	19	$aa = 2709{,}125 - 2547{,}125 = 162$
10ω^2	20	$a = \sqrt{162} = 12{,}72$ &c.
Then	21	$e = \dfrac{162 - 1305}{12{,}72}$, which is impoſſible.
Therefore		$a = 72{,}5$ } as at the 17th and 18th Steps.
And		$e = 54{,}5$

This Queſtion may be performed with leſs Trouble, by ſubſtituting Letters for the known Numbers.

Viz. $\begin{cases} aa + ee = z \\ ae + ee = p \end{cases}$ Then let, $z - p = d = aa - ae$, &c.

Question 23. It is required to find three such Numbers, that the Sum of the first and second being multiplied with the third, may be 37824; and the Sum of the second and third, multiplied with the first, may be 59944; also, that the Sum of the first and third, being multiplied with the second, may be 52456.

Let a, e, y represent the three Numbers.

Then	1	$ay+ey=37824=b$
	2	$ea+ya=59944=c$ Quære a, e, y.
	3	$ae+ye=52456=d$
$1+2+3$	4	$2ae+2ay+2ye=b+c+d$
Let	5	$z=b+c+d$
$4\div\overline{2}$	6	$ae+ay+ye=\frac{1}{2}z=\dfrac{b+c+d}{2}$
$6-3$	7	$ay=\frac{1}{2}z-d=\dfrac{z-2d}{2}$
$7\div a$	8	$y=\dfrac{z-2d}{2a}$
$6-2$	9	$ye=\frac{1}{2}z-c=\dfrac{z-2c}{2}$
$6-1$	10	$ae=\frac{1}{2}z-b=\dfrac{z-2b}{2}$
$10\div a$	11	$e=\dfrac{z-2b}{2a}$
8×11	12	$ye=\dfrac{z-2d}{2a}\times\dfrac{z-2b}{2a}=\dfrac{zz-2dz-2bz+4bd}{4aa}$
9 and 12	13	$\dfrac{z-2c}{2}=\dfrac{zz-2dz-2bz+4bd}{4aa}$
$13\times 4aa$	14	$2zaa-4caa=zz-2dz-2bz+4bd$
$14\div$	15	$aa=\dfrac{zz-2dz-2bz+4bd}{2z-4c}=55696$
$15\,\omega\,2$	16	$a=\sqrt{55696}=236$
11	17	$e=\dfrac{z-2b}{2a}=158$
8	18	$y=\dfrac{z-2d}{2a}=96$

Question 24. It is required to find two such Numbers, that their Sum being substracted from the Sum of their Squares, may leave 14; and if their Product be added to their Sum, it may make 14.

Let a and e be put for the Numbers, and let $y=a+e$

Then	1	$aa+ee-y=14$ by the Question.
	2	$ae+y=14$

$1+y$

$1+y$	3	$aa+ee=14+y$
$2-y$	4	$ae=14-y$
4×2	5	$2ae=28-2y$
$3+5$	6	$aa+2ae+ee=42-y$
$6ɯ^{2}$	7	$a+e=\sqrt{42-y}$
But	8	$a+e=y$, by Substitution above.
7 and 8	9	$y=\sqrt{42-y}$
$9\circledcirc^{2}$	10	$yy=42-y$
$10+y$	11	$yy+y=42$
11C☐	12	$yy+y+\frac{1}{4}=42+\frac{1}{4}=42{,}25$
$12ɯ^{2}$	13	$y+\frac{1}{2}=\sqrt{42{,}25}=6{,}5$
$13-\frac{1}{2}$	14	$y=6{,}5-\frac{1}{2}=6$
Consequent	15	$a+e=6$, by Restitution from above.
3 and 14	16	$aa+ee=14+6=20$
5 and 15	17	$2ae=28-12=16$
$16-17$	18	$aa-2ae+ee=4$
$18ɯ^{2}$	19	$a-e=\sqrt{4}=2$
$15+19$	20	$2a=8$
$20\div 2$	21	$a=4$
-21	22	$e=6-4=2$

Proof $\left\{\begin{array}{r} \text{If } a=4, \text{ and } e=2 \\ \text{Then } aa+ee-a-e=14 \\ \text{And } ae+a+e=14 \\ \text{According to the Questiou.} \end{array}\right.$

Question 25. Three Men discoursing of their Money; saith the first, if 100*l.* were added to my Money, it would be as much as both your Money put together; saith the second Man, if 100*l.* were added to my Money, I should have twice as much as both you have; saith the third Man, if 100*l.* were added to my Money, I should have then three times as much Money as both you have: How much Money had each Man?

Let *a* represent the first Man's Money, *e* the second, and *y* the third.

Then $\left\{\begin{array}{rl} 1 & a+100=e+y \\ 2 & e+100=2a+2y \\ 3 & y+100=3a+3e \end{array}\right\}$ by the Question

$1-a$	4	$e+y-a=100=s$
$2-e$	5	$2a+2y-e=100=s$
$3-y$	6	$3a+3e-y=100=s$
4 and 6	7	$e+y-a=3a+3e-y$
$7+$	8	$2y=4a+2e$
$5-\overline{8}$	9	$2a-e=s-4a-2e$
$9+4a-2e$	10	$6a+e=s=100$
$4+6$	11	$2a+4e=2s=200$

Quære a, e, y,

Ff 2 10×4

10×4	12	$24a+4e=4s=400$
$12-11$	13	$22a=2s=200$
$13 \div 22$	14	$a=\frac{s}{11}=\frac{100}{11}=9\frac{1}{11}l.$
$10-6a$	15	$e=s-6a=100-\frac{600}{11}=\frac{500}{11}=45\frac{5}{11}l.$
$8 \div 2$	16	$y=2a+e=\frac{200}{11}+\frac{500}{11}=\frac{700}{11}=63\frac{7}{11}l.$

Anſwer. The $\left\{\begin{matrix}\text{firſt}\\ \text{ſecond}\\ \text{third}\end{matrix}\right\}$ Man had $\left\{\begin{matrix}9l. & 1s. & 9\frac{9}{11}d.\\ 45l. & 9s. & 1\frac{1}{11}d.\\ 64l. & 12s. & 8\frac{1}{11}d.\end{matrix}\right.$

Queſtion 26. Three Men have each ſuch a Sum of Money, that if the firſt and ſecond Mens Money be added to Half of what the third Man hath; that Sum will be 92*l.* And if the ſecond and third Mens Money be added to one third Part of the firſt Man's Money, that Sum will be 92*l.* Laſtly, if one fourth Part of the ſecond Man's Money be added to the firſt and third Mens Money, that Sum will alſo be 92*l.* How much was each Man's Money?

Put a for the 1*ſt* Man's Money, e for the 2*d*, and y for the 3*d*.

Then	1	$a+e+\frac{1}{2}y=s$
	2	$\frac{1}{3}a+e+y=s$ by the Queſtion; *and* $s=92$
	3	$\frac{1}{4}e+a+y=s$
1 and 2	4	$a+e+\frac{1}{2}y=\frac{1}{3}a+e+y$
$4-e$	5	$a+\frac{1}{2}y=\frac{1}{3}a+y$
$5 \times 2 \times 3$	6	$6a+3y=2a+6y$
$6+$	7	$4a=3y$
2×3	8	$a+3e+3y=3s$
$8-7$	9	$a+3e=3s-4a$
$9-a$	10	$3e=3s-5a$
$10 \div 3$	11	$e=\frac{3s-5a}{3}$
3×4	12	$e+4a+4y=4s=368$
$12-2$	13	$3\frac{2}{3}a+3y=3s=276$
13 and 7	14	$3\frac{2}{3}a+4a=3s=276$
14×3	15	$11a+12a=9s=828$
$15 \div 23$	16	$a=\frac{9s}{23}=\frac{828}{23}=36l.$ the 1*ſt* Man's Money.
11,	17	$e=\frac{3s-5a}{3}=\frac{96}{3}=32l.$ the 2*d* Man's Money.
$7 \div 3$	18	$y=\frac{4a}{3}=\frac{144}{3}=48l.$ the 3*d.* Man's Money.

Queſtion

Queſtion 27. Four Men walking abroad, found a Purſe of Shillings only, out of which every one took a Number at an Adventure; afterwards by comparing their Numbers together they found, that if the firſt took 25 Shillings from the ſecond, it would make his Number equal with what the ſecond had then left; if the ſecond took 30 Shillings from the third, his Money would then be triple to what the third had left, and if the third took 40 Shillings from the fourth, his Money would then be double to what the fourth had left; laſtly, the fourth taking 50 Shillings from the firſt, he would then have three times as much as the firſt had left, and 5 Shillings more: It is required to tell how many Shillings each Man had.

Put a for the firſt Sum, e the ſecond, y the third, and u the fourth.

Then	1	$a+25=e-25$	by the Queſtion.
	2	$e+30=3y-90$	
	3	$y+40=2u-80$	
	4	$u+50=3a-145$	
$1+25$	5	$a+50=e$	
$2-30$	6	$3y-120=e$	
5 and 6	7	$a+50=3y-120$	
$7+120$	8	$a+170=3y$	
$8\div3$	9	$y=\frac{a+170}{3}$	
$3-40$	10	$y=2u-120$	
9 and 10	11	$2u-120=\frac{a+170}{3}$	
$1+120$	12	$2u=\frac{a+170}{3}+120=\frac{a+530}{3}$	
$12\div2$	13	$u=\frac{a+530}{6}$	
$4-50$	14	$u=3a-195$	
13 and 14	15	$3a-195=\frac{a+530}{6}$	
15×6	16	$18a-1170=a+530$	
$16+$	17	$17a=1700$	
$17\div17$	18	$a=100$ the 1*ſt*	Man's Number of Shillings.
by the 5	19	$e=150$ 2*d*	
by the 9	20	$y=90$ 3*d*	
by the 14	21	$u=105$ 4*th*	

Queſtion

Question 28. Four Men had each a Sum of Money, which being put all together makes 250 Pounds; and if to the first Man's Money be added 8 Pounds, it will be just as much as the second Man's Money decreased by 8 Pounds, and as much as 8 times the third Man's Money, and but as much as one eighth Part of the fourth Man's Money; how much had each Man?

Let a, e, y, u, represent the four Mens Money.

Then $\left\{\begin{array}{l|l} 1 & a+e+y+u=s \\ 2 & a+b=e-b \\ 3 & yb=\frac{u}{b}=a+b \end{array}\right\}$ by the Question. Let $s = 250$ and $b=8$, or any other Number at Pleasure.

$2+b$	4	$a+2b=e$
$3\div b$	5	$y=\frac{a+b}{b}$, because $yb=a+b$
$3\times b$	6	$u=ba+bb$, for $\frac{u}{b}=a+b$
$4+5+6$	7	$e+y+u=a+2b+\frac{a+b}{b}+ba+bb$
$1-a$	8	$e+y+u=s-a$
7 and 8	9	$a+2b+\frac{a+b}{b}+ba+bb=s-a$
$9\times b$	10	$ba+2bb+a+b+bba+bbb=bs-ba$
$10+$	11	$2ba+bba+a=bs-bbb-2bb-b$
$11\div$	12	$a=\frac{bs-bb-2bb-b}{bb+2b+1}=16,691358$ &c.
by the 4,	13	$e=a+2b=32,691358$ &c.
by the 5,	14	$y=\frac{a+b}{b}=3,086419$ &c.
by the 6,	15	$u=ba+bb=197,530864$ &c.

That is,

	l.	*s.*	*d.*
$a=$	16	13	9,92592
$e=$	32	13	9,92592
$y=$	3	1	8,74056
$u=$	197	10	7,40739
Consequently $a+e+y+u=$	249	19	11,99976

which should be just 250*l.* the Sum proposed in the Question. Now what it wants of that Sum proceeds from the Imperfection of the Decimal Parts being not continued on to more Places, which would have brought it nearer the Truth, tho' not perhaps exactly so. *Sect.* 5. *Chap.* 5. *Part* I.

Question

Queſtion 29. Several Merchants enter into Partnerſhip, every one put into the Stock 65 times as many Pounds as they were Partners; with that Stock they traded and gained as many Pounds per 100*l.* as they were Partners. Now if 10*l.* 10*s.* be added to, and ſubſtracted from, their Gain, the Product of that Sum and Difference will be 6491*l.* 6*s.* 3*d.*

Quære, How many Merchants there were, *&c.*

Let	1	$a=$ the Number of Merchants.
1×65	2	$65a=$ every one's Sum put into Stock.
$2\times a$	3	$65aa=$the whole Stock.
And	4	$100 : a :: 65aa : \frac{65aaa}{100}$, by the Queſtion.
Viz.	5	$\frac{65\,aaa}{100}=$ the whole Gain.
$5+10,5$	6	$\frac{65\,aaa}{100}+10,5$
$5-10,5$	7	$\frac{65aaa}{103}-10,5$
6×7	8	$\frac{4225\,aaaaaa}{10000}-110,25=6491,3125$, by Queſt.
8×10000	9	$4225a^6-1102500=64913125$
$9+$	10	$4225a^6=66105625$
$10\div 4225$	11	$a^6=\frac{66105625}{4225}=15625$
11ɯ^6	12	$a=\sqrt{}15625=5$ the Number of Merchants.
12×65	13	$65a=325$ the Number of Pounds each put in.

Queſtion 30. Three Merchants join Stock together; the firſt Man's Stock was leſs than the ſecond Man's by 13*l.* the ſecond and third Man's Stock was 175*l.* in trading they gain 48*l.* more than their whole Stock was; the firſt Man's proportional Part of the Gain was 78. What was each Man's Stock and Part of the Gain?

Let *a*, *e*, *y*, repreſent each Man's Stock.

Then	1	$a+e+y=s$ the whole Stock.
	2	$s+48=$ the whole Gain.
And	3	$a+13=e$ } by the Queſtion.
	4	$e+y=175$
$4+a$	5	$a+e+y=175+a$
1 and 5	6	$s=175+a$

6 and 2

6 and 2	7	$s+48=223+a$
But	8	$175+a : 223+a :: a : 78$ per Queſtion.
8 ∵	9	$aa+223a=78a+13650$
9—78a	10	$aa+145a=13650$
10 C □	11	$aa+145a+5256,25=18906,25$
11 w^2	12	$a+72,5=\sqrt{18906,25}=137,5$
12—72,5	13	$a=137,5-72,5=65$
3,	14	$e=a+13=78$
4—14	15	$y=97$
1 Then	16	$65 : 78 :: 78 :$ 93*l.* 12*s.*$=e$'s Gain.
Again	17	$65 : 78 :: 97 :$ 116*l.* 8*s.*$=y$'s Gain.
Proof {	18	116*l.* 8*s.*+93*l.* 12*s.*+78*l*=288*l.* the Gain.
	19	$65+78+97=240$, the whole Stock.
18—19	20	$288-240=48$ the Gain more than the Stock.

Queſtion 31. A Father at his Death left his three Sons his Money in this Manner; to the eldeſt he gave half of it, wanting 44 Pounds; to the ſecond he gave one third of it, and 14 Pounds more; to the youngeſt he gave the Remainder, which was leſs than the Share of the ſecond Son, by 82 Pounds: What was each Son's Share?

Let a, e, y, be the three Shares, and $z=$ the whole Sum?

Then {	1	$a+e+y=z$ } by the Queſtion.
	2	$a=\frac{1}{2}z-44$
	3	$e=\frac{1}{3}z+14$
	4	$y=\frac{1}{3}z+14-82$
2+3+4	5	$a+e+y=\frac{2z}{3}+\frac{z}{2}-98$
1 and 5	6	$z=\frac{2z}{3}+\frac{z}{2}-98$
$6\times\overline{3}$	7	$3z=2z+\frac{3z}{2}-294$
$7\times\overline{2}$	8	$6z=4z+3z-588$
8+	9	$z=588$, the whole Sum that was left.
2 and 9	10	$a=\frac{588}{2}-44=250$. the eldeſt Son's Share.
3 and 9	11	$e=\frac{588}{3}+14=210$, the ſecond Son's Share.
4 and 9	12	$y=\frac{588}{3}+14-82=128$, the youngeſt, &c.

Queſtion 32. A Man playing at Hazard or Dice, won the firſt Throw juſt ſo much Money as he had in his Pocket; the ſecond

ſecond Throw he won the Square Root of what he then had, and five Shillings more; the third Throw he won the Square of all he then had; after which his whole Sum was 112*l.* 16*s.* What Money had he when he began to play?

Suppoſe	1	$a=$ his firſt Sum. Then
1×2	2	$2a=$ his Sum after the firſt Throw.
And	3	$5+\sqrt{2a}=$ the Winnings at the 2*d* Throw.
$2+3$	4	$2a+5+\sqrt{2a}=$ the Sum after the 2*d* Throw.
$2 \circledcirc^{2}$	5	$4aa+22a+25+4a\sqrt{2a}:+10\sqrt{2a}=$ the Winnings at the 3*d* Throw: and therefore
$4+5$	6	$4aa+24a+30+4a\sqrt{2a}+11\sqrt{2a}=2256$ Shill.

But to avoid theſe Surd Quantities, let us, inſtead of ſuppoſing $a=$ the firſt Sum, make a ſecond Trial, *viz.*

Let	1	$2aa=$ the firſt Sum.
1×2	2	$4aa=$ the Sum after the firſt Throw.
Then	3	$2a+5=$ the Sum won at the 2*d* Throw.
$2+3$	4	$4aa+2a+5=$ his Sum after the 2*d* Throw.
$4 \circledcirc^{2}$	5	$16a^4+16a^3+44aa+20a+25=$ the Winnings at the 3*d* Throw; and therefore
$4+5$	6	$16a^4+16a^3+48aa+22a+30=2256$ Shill.

Yet again, to avoid theſe high Equations, let us make a third Suppoſition; thus,

Let	1	$\frac{aa}{2}=$ the firſt Sum.
1×2	2	$aa=$ the Sum after the firſt Throw.
Then	3	$a+5=$ the Winnings at the 2*d* Throw.
$2+3$	4	$aa+a+5=$ the Sum after the 2*d* Throw.
Subſtit.	5	$e=aa+a=5$.
$5 \circledcirc^{2}$	6	$ee=$ the Winnings at the 3*d* Throw. Then
$5+6$	6	$ee+e=2256$ Shillings by the Queſtion.
7C□	8	$ee+e+0,25=2256,25$
8ɯ 6	9	$e+05=\sqrt{2256,25}=47,5$
$9-0,5$	10	$e=47$
5 and 10	11	$aa+a+5=47$
$11-5$	12	$aa+a=42$
12, C□	13	$aa+a+0,25=42,25$
1ɯ 2	14	$a+0,5=\sqrt{42,25}=6,5$
$14-0,5$	15	$a=6$
$15 \circledcirc^{2}$	16	$aa=36$
$16 \div 2$	17	$\frac{aa}{2}=\frac{36}{3}=18$ } The Shillings he had in his Pocket when he began to play.

Note,

Note, In resolving of the last Question, I have made three different Suppositions for the Thing sought, purely as an Instance, to shew the young Learner how well he ought to consider the Nature of the Question, when he first states it, and make choice of representing the Thing sought, so as to avoid running it into Surds, if possible, *viz.* as in the first Supposition of $a =$ the first Sum, *&c.* Not but that such Equations may be solved as shall be shewed in the next Chapter. However, it is most like an Artist to perform Things of this Nature the nearest and easiest Way they can be done.

Question 33. *Suppose there were two equal Circles, whose Peripheries* (viz *Circumferences*) *are divided into* 44310 *equal Parts; and that those Circles were so placed upon one Axis, as to move the contrary Way to each other; and suppose one of them to move but one of these equal Parts the first Day, two Parts the second Day, three Parts the third Day, and so on in Arithmetical Progression,* viz, 1, 2, 3, 4, 5, &c. *and the other to move every Day the Cube of those Parts,* 1, 8, 27, 64, 125, &c. *of the same Parts; How many Parts and how many Days must each Circle move, before the same two Points meet that were together when they began to move?*

In order to give a ready Solution to this Question (or any other in this Kind) it will be convenient to premise this *Lemma.*

LEMMA.

The Sum of any Series of Cubes whose Roots are in Arithmetick Progression (the first Term, and common Difference being Unity or 1) is equal to the Square of the Sum of all those Roots.

As in these

Terms in Arith. *&c.*	Their Cubes.
1	1
2	8
3	27
4	64
5	125
6	216 &c.
—	—

$21 \times 21 = 441$ Sum of their Cubes.

Let	1	$a =$ the Sum of all the Parts the 1*st* Circle moves.	
Then	2	$aa =$ the Sum of all the Parts the 2*d* moves.	
Consequen.	3	$aa + a = 44310$ by the Quest.	(*per Lem.*
2 C ☐	4	$aa + a + 0,25 = 44310,25$	

4*w* [2]

$4uw^2$	5	$a+0,5=\sqrt{44310,25}=210,5$	
$5-0,5$	6	$a=212$	the Number of Parts the first Circle must move.
$6uw^2$	7	$aa=44100$	the Number of Parts the second Circle moves.

Next to find the Number of Days they moved; there is given the first Term $=1$, the common Difference $=1$, and the Sum of all the Terms $=210$, thence to find the last Term, which in this Case is the same with the Number of all the Terms.

Let $a=1$ the first Term, $e=1$ the common Difference, and $s=210$ the Sum all the Terms, to find $y=$ the last Term; as per Sect. 1. Chap. 6. Then $yy+ey=2s+aa-ae$ by the 16 Step, Page 186. that is, $yy+y=210\times2=420$, &c. Hence $y=20$ the Number of Days required.

I shall now proceed to give an Example or two of the Methods used in arguing about unlimited Questions; *viz.* such Questions which admit of various Answers, such as those in *Alligation Alternate* promised in Page 117.

In order to shorten that Work, it will be convenient for the Learner to know the two Signs of Comparison, $\urcorner$ and $\angle$. The Sign $\urcorner$ is of GREATER THAN; as $b \urcorner a$ signifies that b is greater than a. The Sign $\angle$ is of LESSER THAN; as $b \angle d$ signifies that b is lesser than d, &c.

EXAMPLE 1.

Question 34. *A Tobacconist hath three Sorts of Tobacco,* viz. *one of* 2s. 8d. *the Pound, another of* 20d. *the Pound, and a third Sort of* 16d. *the Pound; of these he would make a Mixture to contain* 56 *Pound, that may be sold for* 22d. *the Pound: How much of each Sort may he take?*

Let $a=$ the Quantity of that worth 32 Pence the Pound, $e=$ that of 20 Pence the Pound, and $y=$ that of 16 Pence the Pound;

Then $a+e+y=56$
And $32a+20e+16y=1232$ } *viz.* each Quantity multiplied into its own Price, equals their Sum multiplied into the mean Price.

This

This Queſtion being thus ſtated, it appears by Rule 1, Page 176, that it is capable of innumerable Anſwers; becauſe for any one of theſe three Letters, *a*. *e*, *y*, there may be taken any Number at Pleaſure, provided it be leſs than 56. But although that may be truly done, yet there are ſeveral Ways of arguing about theſe Sorts of Queſtions, which will limit or bound them to all their proper or poſſible Anſwers in whole Numbers. Thus,

Let	1	$a+e+y=56$ } as above.
And	2	$32a+20e+16y=1232$ }
$1-a$	3	$e+y=56-a$
$2-32a$	4	$20e+16y=1232-32a$
3×16	5	$16e+16y=896-16a$
$4-5$	6	$4e=336-16a$
$6\div 4$	7	$e=84-4a$; hence $a < 21$
$3-7$	8	$y=3a-98$; hence $a > \frac{28}{3}$ or $9\frac{1}{3}$

From the two laſt Steps it appears, that the Quantity ſignified by *a*, ought to be leſs than 21, and greater than $9\frac{1}{3}$; that is, any Number betwixt $9\frac{1}{3}$ and 21, may be taken for the Value of *a*: Conſequently there may be eleven Anſwers to this Queſtion in whole Numbers.

Suppoſe $a=10$, then $e=84-40=44$, per 7th Step; and $y=30-28=2$, per 8th Step. Again, if $a=11$, then $e=84-44=40$, per 7th Step, and $y=33-28=5$, per 8th Step; and ſo on for the reſt, which will be as in the following Table.

a	*e*	*y*	*a*	*e*	*y*	*a*	*e*	*y*
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

Thus it will be eaſy to find out and collect all the limited Anſwers to any Queſtion (of this Kind) wherein there are only three Quantities propoſed to be mixed: but when there are more than three, then the Work requires a little more Trouble; becauſe the ſingle Limits of all the Quantities above two muſt be found; that is, if there are four Quantities concerned in the Queſtion, the Limits of two of them muſt be found; if five Quantities are concerned, then the Limits of three of them muſt be found, &c. As in the following Queſtion.

Queſtion

Queſtion 35. *Suppoſe it were required to mix four Sorts of Wines together*; viz. *one Sort worth* 7s. 4d. *the Gallon, another Sort worth* 4s. 7d. *the Gallon, a third Sort worth* 3s. 8d. *the Gallon, and a fourth Sort worth* 2s. 9d. *the Gallon: How much of each Sort may be taken to make a Mixture of* 63 *Gallons, ſo as that the whole Quantity may be ſold for* 5s. 6d. *the Gallon, without Loſs*, &c.

Firſt, let all theſe ſeveral Rates, and the mean Rate, be reduced to one Denomination, *viz.* into Pence.

Viz. $\left\{\begin{array}{l} 7s.\ 4d. = 88d.\ 4s.\ 7d. = 55\,d. \\ 3s.\ 8d. = 44d.\ 2s.\ 9d. = 33\,d. \end{array}\right\}$ and $5s.\ 6d. = 66$.

Put $a =$ the Quantity of that worth 88*d.* the Gallon; $e =$ that of 55*d.* the Gallon, $y =$ that of 44d. the Gallon, and $u =$ that of 33*d.* the Gallon.

Then	1	$a+e+y+u=63$ by rhe Queſtion.
And	2	$88a+55e+44y+33u=4158=63\times66$
$1-a$	3	$e+y+u=63-a$
$-88a$	4	$55e+44+y+33u=4158-88a$
$3\times\overline{33}$	5	$33e+33y+33u=2079-33a$
$4-5$	6	$22e+11y=2079-55a$
$6\div\overline{11}$	7	$2e+y=189-5a$; hence $a \angle \frac{189}{5}$ or $37\frac{4}{5}$
3×55	8	$55e+55y+55u=3465-55a$
$8-4$	9	$11y+22u=33a-693$
$9\div\overline{11}$	10	$y+2u=3a-63$; hence $a \urcorner \frac{63}{3}$ or 21.

From the 7th and 10th Steps it appears, that the Quantity of that Sort of Wine denoted by a, muſt be leſs than $37\frac{4}{5}$ Gallons, and greater than 21 Gallons: that is, it may be $a =$ any Number of Gallons betwixt 21 and $37\frac{4}{5}$. Whence it follows, that there may be collected 16 Anſwers to this Queſtion from the Limits of a only.

Next to find the Limits of e, y, and u.

Suppoſe	11	$a=22$, then will $5a=110$. and $3a=66$
But	12	$2e+y=189-5a=79$, per 7th Step.
$12-2e$	13	$y=79-2e$ hence $e \angle \frac{79}{2}$ or $39\frac{1}{2}$
Again	14	$e+y+u=63-a=41$, per 3d Step.
$14-e$	15	$y+u=41-e$
$15-13$	16	$u=e-38$; hence $e \urcorner 38$

From the 13th and 16th Steps it appears, that if $a = 22$, then $e=39$, $y=79-2e=1$. and $u=e-38=1$.

Again

Again,

Suppose	17	$a=23$, then $5a=115$, and $3a=69$
But	18	$2e+y=189-5a=74$ per 7th Step.
$18-2e$	19	$y=74-2e$; hence $e \angle \frac{74}{2}=37$
Again	20	$e+y+u=63-a=40$, per 3d Step.
$20-e$	21	$y+u=40-e$
$21-19$	22	$u=e-34$, hence $e \urcorner 34$.

From the 19th and 22d Steps it appears, that if $a=13$, then e may be either 35 or 36.

Once more for a further Illustration.

Let	23	$a=24$, then $5a=120$, and $3a=72$
But	24	$2e+y=189-5a=69$, per 7th Step.
$24-2e$	25	$y=69-2e$, hence $e \angle \frac{69}{2}$ or $34\frac{1}{2}$
Again	26	$e+y+u=63-a=39$, per 3d Step.
$26-e$	27	$y+u=39-e$
$27-25$	28	$u=e-30$, hence $e \urcorner 24$

From hence it appears, that if $a=24$, then e may be either 31, 32, 33, or 34, *viz.* it may be any Number betwixt 30 and $34\frac{1}{2}$ by the 25th and 28th Steps; from whence the Value of y and u may be easily found.

$$\text{That is, if} \begin{cases} e=31\,. & \text{then } y=7\,. & \text{And } u=1 \\ e=32\,. & y=5\,. & u=2 \\ e=33\,. & y=3\,. & u=3 \\ e=34\,. & y=1\,. & u=4 \end{cases}$$

Proceeding on in this Manner with all the other single Values of a, there may be found above 120 Answers to this Question in whole Numbers: and if you please to put $a=$ Fractions, there may be found an innumerable Set of Answers; whereas the Rule of *Alligation* in *Vulgar Arithmetick* affords but only one Answer in Fractions, to wit, that of $a=31\frac{1}{2}$, $e=10\frac{1}{2}$, $y=10\frac{1}{2}$, $u=10\frac{1}{2}$; as may be easily tried per Rule *Page* 115, &c.

These two Examples being well understood (especially if the last be thoroughly pursued may suffice to shew the Method of limiting the Answers to all Sorts of Questions of this Kind. I shall therefore conclude this Chapter of Questions with giving a a Solution to the Enigma (or Riddle) proposed (but not answered) Mr. by *John Kersey*, in the Close of the *Appendix* to his *Arithmetic*, which

which affords ſeveral pretty Queſtions, the Solution whereof will diſcover a certain Sentence conſiſting of three Words, which muſt be found by the help of Figures placed (or ſuppoſed to be placed) over the twenty-four Letters of the Alphabet.

Thus $\left\{\begin{array}{l} 1\ .\ 2\ .\ 3\ .\ 4\ .\ 5\ .\ 6\ .\ 7\ .\ \text{\&c.. called Indices.} \\ a\ .\ b\ .\ c\ .\ d\ .\ e\ .\ f\ .\ g\ .\ \text{\&c. to the laſt Letter.} \end{array}\right.$

So that if the Index to that Letter be once found, the letter to which it belongs is conſequently known.

The Enigma.

1. If the Difference between the Indices of the ſecond Letter of the ſecond Word, and the third Letter of the firſt Word, be multiplied into the Difference of their Squares, the Product will be 576; and if their Sum be multiplied into the Sum of their Squares, that Product will be 2336, the Index of the ſaid third Letter being the greateſt.

Let	1	$a=$ the greater Index, or that of the 3d Letter.
And	2	$e=$ the leſſer, or that of the 2d Letter.
Then	3	$\overline{a-e}\times\overline{aa-ee}=576$ } by the Queſtion.
	4	$\overline{a+e}\times\overline{aa+ee}=2336$
3×	5	$aaa-aae-aee+eee=576$
4×	6	$aaa+aae+aee+eee=2336$
6−5	7	$2aae+2aee=1760$
6+7	8	$aaa+3aae+3aee+eee=4096$
6ws³	9	$a+e=\sqrt[3]{4096}=16$
4÷a+e	10	$aa+ee=\frac{2336}{a+e}=\frac{2336}{16}=146$
9 Ⓖ²	11	$aa+2ae+ee=256$
11−10	12	$2ae=110$
10−12	13	$aa=2ae+ee=36$
13ws²	14	$a-e=\sqrt{36}=6$
9+14	15	$2a=22$
15÷2	16	$a=11$
9−16	17	$e=5$

From hence it appears, that the 3d *Letter of the* 1ſt *Word is* l, *and the* 2d *Letter of the* 2d *Word is* e.

Note, *In order to ſet down the Letters (as they become found) in their proper Places, it may be found convenient to ſupply the vacant Places with Stars.*

	First Word.	Second Word.	Third Word.
Thus	* * *l* * *	* *e* * * *	* * * * *

2. The

2. The Indices laſt found, are the two Extreams of four Numbers in Arithmetical Progreſſion, the leſſer Mean being the Index of the firſt Letter of the third Word; and the greater Mean is the Index of the fourth and laſt Letter of the firſt Word. *Viz.* 5 . 7 . 9 . 11 are the four Terms in Arithmetical Progreſſion. Whence it appears, that *G* (whoſe Index is 7) is the firſt Letter of the third Word; and that *i* (whoſe Index is 9) is the fourth or laſt Letter of the firſt Word; which being placed down, will ſtand thus,

** * li * e * * * G * * *.*

3. The ſecond Letter of the third Word is the ſame with the third Letter of the firſt Word; and the fifth Letter of the third Word is the ſame with the laſt Letter of the firſt Word: whence the Letters will ſtand thus,

** * li * e * * * Gl * * i * *.*

4. The Sum of the Squares of the Indices of the firſt and ſecond Letters of the firſt Word is 520, and the Product of the ſame Indices is ſeven Ninths of the Square of the greater Index, which is the Index of the ſaid firſt Letter.

Let $a=$ the greater, and $e=$ the leſſer Index.

Then	1	$aa+ee=520$	according to the *Data*.
And	2	$ae=\frac{7}{9}aa$	
$2\div a$	3	$e=\frac{7}{9}a$	
$3 \circledcirc^{2}$	4	$ee=\frac{49}{81}aa$	
$1-4$	5	$aa=520-\frac{49}{81}aa$	
5×81	6	$81aa=42120-49aa$	
$6+49aa$	7	$130aa=42120$	
$7\div 130$	8	$aa=\frac{42120}{130}=324$	
$8\omega^{2}$	9	$a=\sqrt{324}=18$, whoſe Letter is *s*.	
3 and 9	10	$e=\frac{7}{9}a=14$, whoſe Letter is *o*.	

Hence the Letters will ſtand thus,

*Soli * e * * * Gl * * i **

5. The Difference between the two laſt Indices, is the Index of the firſt Letter of the ſecond Word, *viz* $18-14=4$ being the Index of the Letter *D*. Then the Letters will ſtand thus,

*Soli De * * * Gl * * i *.*

6 The

6. The third and laſt Letter of the ſecond Word, alſo the third Letter of the third Word, are the ſame with the ſecond Letter of the firſt Word; hence the Letters will ſtand thus,

*Soli Deo Glo * i **

7. The Sum of the Indices of the fourth Letter of the third Word, and the ſixth or laſt Letter of the ſame Word, being added to their Product is 35; and the Difference of their Squares is 288; the Index of the laſt Letter being the leaſt.

Put $a =$ the greater, and $e =$ the leſſer Index as before.

$$
\begin{array}{r|c|l}
\text{Then} & 1 & ae+a+e=35 \\
\text{And} & 2 & aa-ee=288 \quad \Big\} \text{ by the } Data. \\
1-a & 3 & ae+e=35-a \\
3 \div \overline{a+1} & 4 & e=\dfrac{35-a}{a+1}, \text{ for } e \times \overline{a+1}=ae+e \\
4 \odot^{2} & 5 & ee=\dfrac{1225-70a+aa}{aa+2a=1} \\
2+5 & 6 & aa=288+\dfrac{1225-70a+aa}{aa+2a+1} \\
6 \times aa \text{ \&c.} & 7 & \begin{cases} a^4+2a^3+aa=288aa+576a+288 \\ +1225-70a+aa \end{cases} \\
7+ & 8 & a^4+2a^3-288aa-506a=1513
\end{array}
$$

This laſt Equation being reſolved according to the Method which ſhall be ſhewed in the next Chapter, it will be $a=17$ its Letter; and from the 4th Step $e=\dfrac{35-a}{a+1}=1$, the Index of the Letter *a*. Then theſe two Letters being placed according to the *Data* above, are all that are required by the Enigma to compleat theſe Words,

Soli Deo Gloria.

CHAP. X.

The Solution of ADFECTED EQUATIONS *in Numbers.*

BEFORE we proceed to the Solution of Adfected Equations, it may not be amiss to shew the Investigation (or Invention) of those Theorems or Rules for extracting the Roots of Simple Powers, made use of in Chapter XI. Part I. I shall here make choice of the same Letters to represent the Numbers both given and sought, as in my Compendium of *Algebra*.

G, always denote the given Resolvend.

Viz. Let $r=$ any Number taken as near the true Root as may be, whether it be greater or less.

$e=$ the unknown Part of the Root sought, by which r is to be either increased or decreased.

Then if r be any Number less than the true Root, it will be $r+e=$ the Root sought. But if r be taken greater than the true Root, it will then be $r-e=$ the Root sought. And put D for the Dividend that is produced from G, after it is lessened and divided by r, *&c.* (into the Co-efficients of Adfected Equations) according as the Nature of the Root requires. These Things being premised, we may proceed to raising the Theorems.

SECT. I.

1. FOR the Square Root, *viz.* $aa=G$. Quære a.

Let	1	$r+e=a$
1 ⓦ 2	2	$rr+2re+ee=aa=G$
2 — rr	3	$2re+ee=G-rr$. Call it D, viz. $D=G-rr$.
Then	4	$\dfrac{D}{2r+e}=e$ This shews the 1st Method of extracting the Square Root, Sect. 5. Chap. XI. Part I.
3 ÷ 2	5	$re+\frac{1}{2}ee=\dfrac{G-rr}{2}=D$.

Which gives this Theorem $\dfrac{D}{r+\frac{1}{2}e}=e$.

The Arithmetical Operations of both these Theorems, you have in the Examples of Section 2, Page 126, to which I refer the

the Learner, ſuppoſing him, by this Time, to underſtand them without any more Words than what is there expreſs'd.

II. To extract the Cube Root; *viz.* $aaa=G$. Quære a.

Let	1	$r+e=a$, ſuppoſing r leſs than the true Root.
1 ⓖ. 3	2	$rrr+3rre+3ree+eee=aaa=G$
$2-rrr$	3	$3rre+3ree+eee=G-rrr$
$3\div 3r$	4	$re+ee+\frac{eee}{3r}=\frac{G-rr}{3r}=D$

Let $\frac{eee}{3r}$ be rejected or caſt off, as being of ſmall Value; then it will be, $re+ee=D$, which gives this following

Theorem $\frac{D}{r+}=e$

By this Theorem or Rule, the 1ſt and 2d Examples in Caſe 1. Page 132, are performed; the which being compared with this Theorem may be eaſily underſtood.

Again, Suppoſe $aaa=G$, as before, and let r be taken greater than the true Root.

Then	1	$r-e=a$	eee being rejected as before.
1 ⓖ. 3	2	$rrr-3rre+3rre=a^3=G$	
$2+$	3	$3rre-3ree=rrr-G$	
$3\div 3r$	4	$re-ee=\frac{rrr-G}{3r}$	

Which gives this *Theorem* $\frac{D}{r-e}=e$

By this Theorem the third Example in Caſe 2, Page 133, is performed.

III. To extract the *Biquadrate* Root; *viz.* $a^4=G$, Quære a.

Let	1	$r-e=a$ ſuppoſing r leſs than juſt.	
1 ⓖ. 4	2	$r^4+4rrre+6rree=a^4=G$	rejecting all the Powers of e above ee.
$2-r^4$	3	$4rrre+6rree=G-r^4$	
$3\div 2rr$	4	$2re+3ee=\frac{G-r^4}{2rr}=D$	

Which gives this *Theorem* $\frac{D}{2r+3e}=e$

By this Theorem the Biquadrate Root of any Number may be extracted. But, as I have already said, Page 134, those Extractions may be very well performed by two Extractions of the Square Root. *Vide Example, Page* 135.

IV. To extract the *Surfolid* Root, *viz.* $a^5=G$. Quære a.

If r be taken less than just, then $r+e=a$, as before, and $\frac{G-r^5}{5r^3}=D$, which gives this *Theorem* $\frac{D}{r-2e}=e$. By this Theorem the Surfolid Root, Example 1, Page 136, is extracted. But if r be taken greater than just; then $r-e=a$, and $\frac{r^5-G}{5r^3}=D$, which gives this *Theorem* $\frac{D}{r-2e}=e$. By this last Theorem the Example in Page 137 is performed.

I presume it needless to pursue the raising of those Theorems, for extracting the Roots of simple Powers, any further; because the Method of doing it is general, how high soever they are; and therefore it may be easily understood by what is already done.

SECT. 2.

NOtwithstanding I have already shewed the Solution of Quadratic Equations, two several Ways, *viz.* by casting off the lowest Term; and by compleating the Square, *vide* Section 2, Page 195, &c. Yet it may not be amiss to shew, how those Equations may be resolved into Numbers by this universal Method of continued Series; wherein, if the first r be taken equal to the first true Root, or single Side of the Resolvend; and every single Value of e (as it becomes found) be still added to it, for a new r, then those Roots may be extracted without repeating a second Operation, as before in the single Powers.

Case 1. Let $aa+2ba=G$. It is required to find the Value of a.

Put	1	$r+e=a$
$1 \circlearrowleft^2$	2	$rr+2re+ee=aa$
$1 \times 2b$	3	$2br+2be=2ba$
$2+3$	4	$rr+2br+2re+2be+ee=aa+2ba=G$
$4-rr$ &c.	5	$2re+2be+ee=G-rr-2br$
$5 \div 2$	6	$re+be+\frac{1}{2}ee=\frac{1}{2}G-\frac{1}{2}rr-br=D$

Which gives this *Theorem* $\frac{D}{r+b+\frac{1}{2}e}=e$

Suppose

Suppose $b=364$, and $G=38692865$: If $r=6000$, then $rr=36000000$, and $2br+4361000$. But $36000000+4368000=40368000 \urcorner 38692865=G$. Therefore the first $r \angle 6000$. Let $r=5000$, then

1st $r=5000$	$19346432,5=\frac{1}{2}G$	
$b=\underline{364}$	$-\ 1432000, \ =\frac{1}{2}rr+br$	
1st $r+b=5364$	$5026432,5 = D$	$(800=e$
$+\frac{1}{2}e=\underline{400}$	$\underline{46112}$	
1 Divisor 5764)	41523	$(\ 60=e$
2d $r+b=6164$	$\underline{37164}$	
$+\frac{1}{2}e=\underline{30}$		
	$43592,5$	$(\ \ 7=e$
2 Divisor 6149)	$\underline{43592,5}$	
3d $r+b=6224$	(0)	$867=e$
$+\frac{1}{2}e=\underline{3.5}$		
3 Divisor 6227,6		

$\left.\begin{array}{r}\text{First } r=5000\\ +e=\ 867\end{array}\right\} =5867=a$ as was required.

Case 2. If $aa-2ba=G$, then proceeding as above, there will arise this *Theorem* $\frac{D}{r-b=\frac{1}{2}e}=e$, &c. And in *Case* 3, viz. $2ba-aa=G$, you will have this *Theorem* $\frac{D}{b-r-\frac{1}{2}e}$ &c, as above,

I think it needless to trouble the Reader with the Work of these two Theorems in Numbers; because if the last Example of *Case* 1, be understood, the other will be easy. Not but that the Method of compleating the Square is very ready and easy, as you may observe by the Work in several Questions of this Chapter.

SECT. 3.

IN the Solution of all Adfected Equations, that are above (or higher than) Quadraticks, it will be the best Way to take $r=$ the next nearest Root of the Equation: And then it will be $r+e=a$, if r be less than just; or $r-e=a$ if r be greater than just (as at the beginning of this Chapter). And all the Powers of the unknown Part of the Root, (*viz. e*) above its Square (*ee*) are to be rejected or cast off, as before in raising the Theorems for the Simple

Simple Powers. And therefore it is, that to supply the want of those Powers (above ee in the Theorem) the Operation must be repeated: as in the Example of extracting the Cube Root, Page 133, *viz.* when the Figures in the Root consist of more than three Places (*vide* Page 140, and 141.)

Suppose $aaa+ba=G$. Quære a.

Let	1	$r+e=a$ viz. let r be supposed less than just.
$1\,\circledS^{3}$	2	$rrr+3rre+3ree=aaa$
$1\times b$	3	$br+be=ba$
$2+3$	4	$rrr+br+3rre+be+3re=a^3+ba=Ge$
$4\div 3r$	5	$\frac{1}{3}rr+\frac{1}{3}b+re+\frac{be}{3r}+ee=\frac{G}{3r}$
5—&c.	6	$re+\frac{be}{3r}+ee=\frac{G}{3r}-\frac{1}{3}rr-\frac{1}{3}b=D$

Which gives this *Theorem* $\dfrac{D}{r+\frac{b}{3r}+e}=e$

But if r be taken greater than just, then it will be $re+\frac{be}{3r}$ $-ee=\frac{1}{2}rr+\frac{1}{3}b-\frac{G}{3r}=D$, which produces this *Theorem*

$$\frac{D}{r+\frac{b}{3r}-e}=e$$

By either of these two Theorems the Value of a may be easily found. Or rather otherwise, as in the following Example.

Let $aaa+24a=587914$. Here $b=24$. Suppose the first $r=90$, then $r^3=729000 \urcorner 587914$ without the 24×90 being added to it: Therefore $r \angle 90$. Again, Suppose $r=80$ then $r^3=512000$, and $24\,r=1920$. But $512000+1920=513920 \angle 587914$, hence $\urcorner 80$, but nearer to it than 90. Therefore

it must be	1	$r+e=a$ less than just
$1\,\circledS^{3}$	2	$rrr+3rre+3ree=aaa$
1×24		$24r+24e=24a$
2 in Numb.	4	$512000+19200e+240ee=aaa$
3 in Numb.	5	$1920+\quad 24e=24a$
$4+5$	6	$513920+19224e+240ee=587914$
$6-513920$	7	$19224e+240ee=73994$
$7\div 240$	8	$80,1e+ee=308,31=D$
$8\div$	9	$e=\dfrac{D}{80,1+e}$

Operation

Operation 80,1
$+e=3,$

1 Divifor 83,1)	308,31	80, $=r$ 3,68 &c. $=e$
$+e=$3,6	249,3	83,68 &c. $=r+e$
2 Divifor 86,7)	59,01	
$+e$,67	52,02	
87,37)	6,99 &c.	

Or rather new $r=83,7$ for a fecond Operation, which being involved and tryed (as above) will be found greater than juft: therefore

it muft be	1	$r-e=a$
1 $\mathcal{G}^{3}$	2	$rrr-3rre+3ree=aaa$
1×24	3	$24r-24e=24a$
2 in Numb.	4	$586376,253-21017,07e+251,1\,ee=aaa$
3 in Numb.	5	$2008,8\quad-\quad24e=24a$
4+5	6	$588385,053-21041,07e+251,1\,ee=587914$
6+	7	$21041,07e-251,1ee=471,053$
$7\div251,1$	8	$83,7955e-ee=1,87595778=D$
$8\div$	9	$e=\dfrac{D}{83,7955-e}$

2d Operation 83,7955
$-e=$,02

1ft Divifor 83,7755)	1,87595778	83,70000000 $=r$ 00,02239331 $=e$
$-e=$,022	1,675510	83,67760669 $=a=r-e$.
2d Divifor 83,7535)	,2004477	
$-e=$,0033	,1675070	
3d Divifor 83,75 15)	,03294078	
$-e=$ 5 &c,	,02512536	
83,751	,00781542	
...	00753760	
Here the new Divifors are rejected, as infignificant.	27782	
	25125	
	2657	
	2512	
	145	
	13	

All the remaining Examples of extracting Roots (except Page 260) are left in the Author's own Method; which by this Time, it is prefumed, the Learner will eafily know how to correct himfelf, if he takes due Notice of what has been delivered Page, 131, 132, &c.

But

But if more Exactneſs be required, you may make the new $r=83,6776067$, and proceed with it to a third Operation; which will afford twenty-ſeven Places of Figures for the Value of a; that is, every Operation will produce triple the Places of Figures to thoſe of the precedent r. And this tripling the Places of Figures in the Root, at every Operation, holds good, and is to be obſerved in the Solution of all Adfected Equations (how high ſoever they are) according to this Method of reſolving them. *See Page* 141.

Example 2. Suppoſe $aaa - ba = G$. Quære a. If $r + e = a$, then $re - \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}aa = D$, which gives this Theorem $\frac{D}{r - \frac{\frac{1}{3}b}{r} + e} = e$. But if $r - e = a$, then $re + \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + {}^{1}b - \frac{1}{3}rr$ $= D$, which gives this Theorem $\frac{D}{r + \frac{\frac{1}{3}b}{r} + e} = e$

Or you may proceed otherwiſe, as in the laſt Example. Let $aaa - 6438\,a = 104785688$, here $b = 6438$. Suppoſe the firſt $r = 500$, $rrr = 125000000$, and $br = 3219000$, then $125000000 - 3219000 = 12{,}1781000$. But $121781002 \urcorner 104785688$, therefore $r \angle 500$. Again, ſuppoſe $r = 400$, $rrr = 64000000$, and $br = 2575200$, then will $64000000 - 2575200 - 6142800$. But $61424800 \angle 10475688$, hence $r \urcorner 400$; conſequently r is betwixt 400 and 5000. But 500 is the next neareſt; therefore, let $r = 500$ being greater than juſt.

Then	1	$r - e = a$
1 Ⓖ 2	2	$rrr - 3rre + 3ree = aaa$
$1 \times b$	3	$br - be = ba$
2 in Numb.	4	$125000000 - 750500e + 100\,ee = aaa$
3 in Numb.	5	$3219000 - 6438e = 6438a$
$4 - 5$	6	$121781000 - 743562e + 1500ee = 104785988$
$6 +$	7	$743562 - 1500ee = 16695312$
$7 \div 1500$	8	$495e - ee = 11350 = D$
$8 \div$	9	$e = \frac{D}{495 - e}$

Operation

Operation	495		
$-e=$	20		
1 Divisor	475)	11330(	500,0 $=r$
			23,8 $=e$
$-e=$	3	950	476,2 $=r-e=a$
	472)	1830	
		1416	
		414,0	
		377,6	

Let new $r=476$ for a 2d Operation; then $r^3=107850176$ and $br=3064488$: but $107850176-3064488=104785688$ the ſame with the Reſolvend. Conſequently $a=476$ juſt.

Example 3. Let $ba-aaa=G$. Quære a. If $r+e=a$, then $\frac{\frac{1}{3}be}{r}-re-ee=\frac{\frac{1}{3}G}{r}+\frac{1}{3}rr-\frac{1}{2}b=D$, which gives this *Theorem* $\dfrac{D}{\frac{\frac{1}{3}b}{r}-re}=e$

But if $r-e=a$, then $re-\frac{\frac{1}{3}be}{r}-ee=\frac{\frac{1}{3}G}{r}+\frac{1}{3}rr-\frac{1}{4}b=D$ which gives this *Theorem* $\dfrac{D}{r-\frac{\frac{1}{3}b}{r}-e}=e$.

Or otherwiſe as before in the two laſt Examples. Thus, let $123456a-aaa=12272861$. Here $b=123456$. Suppoſe the firſt $r=200$, then $rrr=8000000$, and $br=24691200$; then $24691200-8000000=16691200$, but $16691200 \urcorner 12272861$, therefore r is here leſs than juſt, becauſe the higheſt Power is—, or Negative. Again, Suppoſe $r=300$, then $r^3=27000000$, and $br=37036800$, then $37036800-27000000=10036800 \angle 12272861$. Conſequenlty $r \angle 300$, and $r \urcorner 200$. Let $r=300$, being the next neareſt, but more than juſt.

Then	1	$r-e=a$
1 ⓖ. 3	2	$rrr-3rre+3ree=aaa$
$1\times b$	3	$br-be=ba$
2 in Numb.	4	$27000000-270000e+900ee$
3 in Numb.	5	$37036800-123456e$
$5-4$	6	$10036800+146544e-900ee=12272861$
$6-$	7	$146544e-900ee=2236061$
$7\div 900$	8	$162e-ee=2484=D$
$1\div$ &c.	9	$e=\dfrac{D}{162-e}$

$$\begin{array}{lll}
\text{Operation } 162 & & \\
\quad -e = \ 10 & & \\
\text{1st Divisor } 152) & 2484 & (300{,}0=r \\
\quad -e = \ \ 6 & 152 & \ \ 16{,}6 = e \\
\text{2d Divisor } 146) & 964 & 283{,}4 = r - e = a. \\
 & 876 & \\
 & 88{,}0 & \\
 & 86{,}6 &
\end{array}$$

Or new $r=283$, which being involved, *&c.* will appear to be the true Root, that is, $a=283$ just.

Note, These are usually called the three Forms of Cubick Equations; and in the Solution of the third or last Form, *viz.* $ba-aaa=G$, you may meet with some seeming Difficulties; especially in making Choice of the first r, because this Equation is an ambigious Equation, and hath two Affirmative Roots, *viz.* a greater and lesser Root. But having once found either of them, the other may be easily obtained by Division only; as in the Quadartick Equation. *Vide* Chap. VIII. As for instance, in the last Example, $a=283$ and $123456\,a-aaa=12272861$. Make these two Equations $=0$, to wit, let $a-283=0$, and $-aaa+123456\,a-12272861=0$.

$$\begin{array}{lrl}
\text{Then, } a-283) & -aaa+123456\,a-12272861 & (-aa \\
 & -aaa+283\,aa & \\
 & -283aa+123456a & (-283a \\
 & -283aa+\ 80089a & \\
 & +43367a-12272861 & (+43367 \\
 & +43367a-12272861 & \\
 & (0) \qquad (0) &
\end{array}$$

Hence it appears that $-aa-283a+43367=0$. Consequently $aa+283a=43367$ this Equation being solved, $a=110{,}2722$ *&c.* which is the lesser Root of the aforesaid Equation $ba-aaa-G$, &c. After this Manner all the possible and impossible Roots of any Equation may be easily discovered, any one of its Roots being once found. I shall therefore omit inserting more Examples of that Kind.

Suppose $aaa+baa+ca=G$. Quære a. Let $b=74$, $c=8729$, and $G=560783$. By Trial (as before) it will be found that the next nearest $r=40$ being something less than just.

Therefore

Therefore	1	$r+e=a$
$1\times c$	2	$cr+ce=a$
$1\,\odot^{2}:\times b$	3	$brr+2bre+bee=baa$
$1\,\odot^{3}$	4	$rrr+3rre+3ree=aaa$
2 in Numb.	5	$394160+\ 8729e$
3 in Numb.	6	$118400+\ 5920e+\ 74ee$
4 in Numb.	7	$64000+\ 4800e+120ee$
$5+6+7$	8	$531560+19449e+194ee=560783$
$3-531560$	9	$19449e+1949\,ee=29223$
$9\div 194$	10	$100{,}2\,e+ee=153{,}06=D$
$10\div$	11	$e=\dfrac{D}{100{,}2+e}$

Operation 100,2
$+e=$ 1
1ſt Diviſor 101,2) 153,06 (40,0 $=r$
                                   1,5 $=e$
$+e=$,5 101,2
2d Diviſor 101,7) 51,86 (41,5 $=r+e=a$
50,85
1,01

Or new $r=41{,}5$ for a ſecond Operation, which being duly involved, &c. will be found more than juſt.

Therefore	1	$r-e=a$
Then	2	$cr-ce=ca$
	3	$brr-2bre+bee=baa$
	4	$rrr-3rre+3ree=aaa$

Theſe being turned into Numbers, &c. as above, they will be $20037{,}75e-198{,}5ee=390{,}375$, which being divided by 198,5 the Co-efficient of ee, will become $100{,}946e-ee=1{,}966624$, &c. $=D$.

Operation 100,946
$-e=$ 01
1ſt Diviſor 100,936) 1,966624 (41,5000000 $=r$
                                         0194847 $=e$
$-e=$,009 1,00936 41,4805153 $=r-e=a$
2d Diviſor 100,927) 957264
908343
*489210
403708
855020
807416
476040
403708
72332 &c

Here I proceed by plain Diviſion without forming new Diviſors.

Let the laſt Equation in the Enigma, Chap. IX. be here propoſed for a Solution. *Viz.* $aaaa+baaa-caa-da=G$; $b=2$, $c=288$, $d=506$, and $G=1513$. Quære a. By Tryals it will be found, that the next neareſt $r=20$, being ſomething more than juſt.

Therefore	1	$r-e=a$
$1\times d$	2	$dr-de=da$
$1\circledcirc^{2}\times c$	3	$crr-2cre+cee=caa$
$1\circledcirc^{3}\times b$	4	$brrr-3brre+3bree=baaa$
$1\circledcirc^{4}$	5	$r^4-4rrre+6rree=aaaa$

Theſe being turned into Numbers, and thoſe duly collected, according as the Signs of the Equation direct, they will become $50680-22374e+2232ee=1513$, which being all divided by 2232 the Co-effiicient of ee, will be $10\,e-ee=22=D$.

$$\text{Then } \frac{D}{19-e}=e$$

Operation 10
$-e=$ 3

Divisor 7) 22 ($20=r$
$3=e$
21 $17=r-e=a$ juſt.
1

See the End of Chap. IX.

By what hath been already done about the Solution of theſe few Equations (being carefully obſerved) I preſume the Learner will eaſily conceive how to proceed in the Solution of all Kinds of Equations, be they never ſo high, or adfected; therefore I ſhall not here promiſe many various Examples, but only take them as they fall in Courſe, when I come to the next Part, wherein you will (perhaps) find ſuch Equations with their Solutions as are not common.

CHAP.

CHAP. XI.

Of Simple INTEREST, ANNUITIES, *or Penſions*, &c.

INTEREST, or the Uſe paid for the Loan of Money, is either Simple; or Compound.

Sect. 1. *Of Simple* INTEREST.

SIMPLE Intereſt, is that which is paid for the Loan of any Principal or Sum of Money, lent out for ſome Time, at any Rate *per Cent.* agreed on between the Borrower and the Lender; which, according to the late Laws of *England*, ought to be ſix Pounds for the Uſe of 100*l.* for one Year, and twelve Pounds for the Uſe of 100*l.* for two Years: and ſo on for a greater, or leſſer Sum, proportionable to the Time propoſed.

There are ſeveral Ways of computing (or anſwering Queſtions about) Simple Intereſt; as by the ſingle and double Rule of Three (See Page 96, &c.) others make uſe of Tables compoſed at ſeveral Rates *per Cent.* as Sir *Samuel Moreland*, in his Doctrine of Intereſt both ſimple and compound, all performed by Tables; wherein he hath detected ſeveral material Errors committed by Sir *Iſaac Newton*, Mr. *Kerſey* upon *Wingate* and Mr. *Clavel.* &c. in the Buſineſs of computing Intereſt, &c. by their Tables too tedious to be here repeated. But I ſhall in this Tract take other Methods, and ſhew that all Computations relating to ſimple Intereſt are grounded upon Arithmetical Progreſſion; and from thence raiſe ſuch general Theorems, as will ſuit with all Caſes. In order to that

$$\text{Let}\left\{\begin{array}{l} P = \text{any Principal or Sum put to Intereſt.} \\ R = \text{the Ratio of the Rate, \textit{per Cent. per Annum}.} \\ t = \text{the Time of the Principal's Continuance at Intereſt.} \\ A = \text{the Amount of the Principal, and its Intereſt.} \end{array}\right.$$

Note, The Ratio of the Rate, is only the ſimple Intereſt of 1*l.* for one Year, at any given Rate; and is thus found.

Viz. 100 : 6 :: 1 : 0,06 = the Ratio at 6 *per Cent. per Annum.*
Or 100 : 7 :: 1 : 0,07 = the Ratio at 7 *per Cent.* &c.
Again 100 : 7,5 :: 1 : 0,075 = the Ratio at 7 and ½ *per Cent.*

And if the given Time be whole Years; then t = the Number of whole Years: but if the Time given, be either pure Parts of a Year, or Parts of a Year mixed with Years; thoſe Parts muſt be turned into Decimals; and then t = thoſe Decimals, &c.

Now

Now the common Parts of a Year may be easily turned or converted into Decimal Parts, if it be considered

That one { Day is the $\frac{1}{365}$ Part of a Year $=0{,}00274$ *ferè*
Month is the $\frac{1}{12}$ Part of a Year $=0{,}0833333$ &c.
Quarter is the $\frac{1}{4}$ Part of a Year $=0{,}25$ }

These Things being premised, we may proceed to raising the *Theorems*.

Let $R=$ the Interest of 1*l.* for one Year, as before.
Then $2R=$ the Interest of 1*l.* for two Years.
And $3R=$ the Interest of 1*l.* for three Years.
$4R=$ the Interest of 1*l.* for four Years. And so on for any Number of Years proposed.

Hence it is plain, that the simple Interest of one Pound is a Series of Terms in Arithmetical Progression increasing; whose first Term and common Difference is R, and the Number of all the Terms is t. Therefore the last Term will always be $tR=$ the Interest of 1*l.* for any given Term signified by t.

Then { *As one Pound : is to the Interest of* 1l. : : *so is any Principal or given Sum : to its Interest.* }

That is, $1l. : tR :: P : tRP=$ the Interest of P. Then the Principal being added to its Interest, their Sum will be $=A$ the Amount required: which gives this general *Theorem*.

Theorem 1. $tRP+P=A$.

From whence the three following *Theorems* are easily deduced.

Theorem 2. $\frac{A}{tR+1}=P$. *Theorem* 3. $\frac{A-P}{tp}=R$.

Theorem 4. $\frac{A-P}{RP}=t$.

These four *Theorems* resolve all Questions about simple Interest.

Question 1. *What will* 256l. 10 s. *amount to in* 3 *Years, one Quarter,* 2 *Months, and* 18 *Days, at* 6 *per Cent. per Annum?*

Here is given $P=256{,}5$; $R=0{,}06$; and $t=3{,}46599$
For 3 Years $=3$ Quære A. per *Theorem.* 1.
one Quarter $=0{,}25$
2 Months $=0{,}16667=0{,}08333\times 2$
18 Days $=0{,}04932=0{,}00274\times 18$

Hence $t=3{,}40599:\times 0{,}06$ $0{,}2079594=tR$
Then $0{,}2079594\times 256{,}5=53{,}341586=tRP$
And $53{,}341586+256{,}5=309{,}841586=tRP+P=A$.
That is, $309{,}841586=309l.\ 16s.\ 10d.$ being the Answer required.

Question

Queſtion 2. *What Paincipal or Sum being put to Intereſt, will raiſe a Stock of* 309l. 16s. 10d. in three Years, one Quarter, two Months, and 18 Days; at 6 *per Cent. per Annum?*

Or the ſame Queſtion otherwiſe ſtated thus.

What is 309l. 16s. 10d. *due three Years, one Quarter, two Months, and* 18 *Days hence, worth in ready Money; abating or diſcounting* 6 *per Cent*, &c.

Here is given $A = 309,841586$; $R = 0,06$; $t = 3,46599$ (fonnd as before) thence to find P. *Per Theorem* 2. Firſt $3,46599 \times 0,06 = 0,2079594 = tR$. Then $tR + 1 = 1,2079594$) $309,841586 = A$ ($256,5 = P$; that is, $256,5 = 256\,l.\ 10\,s.$ the Anſwer required.

Queſtion 3. *At what Rate or Intereſt per Cent.* &c. *will* 256l. 10s. *amount to* 309l 16s. 10d. *in three Years, one Quarter, two Months, and* 18 *Days?*

Here is given, $P = 256,5$; $A = 309,841586$; and $t = 3,46599$ to find R. *Per Theorem* 3. Firſt $309,841586 - 256,5 = 53,341586 = A - P$. Next $3,46599 \times 256,5 = 889,026435 = tR$. And $tR = 889,026435$) $53,341586$ ($00,06 =$ the Ratio. Then $1\,l. : 0,06 :: 120 : 6 =$ the Rate required.

Queſtion 4. *In what Time will* 256l. 10s. *raiſe a Stock of (or amount to)* 309l. 16s. 10d. *at* 6 *per Cent.* &c.

Here is given, $P = 256,5$; $A - 309,841586$, and $R = 0,06$ to find t. *Per Theorem* 4. Firſt $309,841586 - 256,5 = 53,341586 = A - P$. And $256,5 \times 0,06 = 15,39 = PR$. Then $15,39$) $53,341586$ ($3,46599 = t$; that is, $t = 3$ Years and ,46599 Decimal Parts of a Year; which may be brought into common Parts of a Year, thus

0,46599	And 0,08333) 0,21599 (2 Months
0,25 — one Quarter	,16666
0,21599	0,02074) ,04933 . (18 Days.

Hence $t = 3$ Years, one Quarter, 2 Months, and 18 Days; the Anſwer required.

It muſt needs be eaſy to conceive, that what is here done at 6 *per Cent.* may be done at any other Rate of Intereſt, by forming the Ratio *(viz R)* accordingly.

SCHOLIUM.

SCHOLIUM.

Although it be according to the Laws and Cuſtom of *England*, to compute Intereſt at the Proportion of 6 *per Cent.* (as above) yet he that takes up Money at Intereſt for any Time leſs than even or compleat Years, pays more Intereſt than ſeems reaſonably due, according to the Rules of Art. As for Inſtance; if 100*l.* be forborne at Intereſt one whole Year, it amounts to 106*l.* But (I ſay) if it be repaid at the half Year's End it ſhould not amount to 103; as appears from this following Proportion.

Let $a=$ the Amounts due at the half Year's End; then it will be $100 : a :: a : 106$ the Amount at the Year's End. *Ergo* $aa=10600$, and $a=\sqrt{10600}=102,9563=$ 102*l.* 19*s* 1¾*d.* which is leſs than 103*l.* by 10¼*d.* And if it be paid in leſs than half a Year's Time, the Error muſt needs be the greater.

Sect. 2. *Of* Annuities, *or* Pensions *in Arrears, computed at ſimple Intereſt.*

ANNUITIES, or Penſions, &c. are ſaid to be in Arrears, when they are payable or due, either Yearly, or Half-yearly, &c. and are unpaid for any Number of Payments. Therefore the Buſineſs is, to compute what all thoſe Payments will amount unto, allowing any Rate of ſimple Intereſt for their Forbearance, from the Time each particular Payment became due: Now in order to that,

Put $\begin{cases} u = \text{the Annuity, Penſion, or Yearly Rent, \&c.} \\ t = \text{the Time of it's Continuance, or being unpaid.} \\ R = \text{the Ratio, or Intereſt of 1}l.\text{ for 1 Year, as before.} \\ A = \text{the Amount of the Annuity and its Intereſt.} \end{cases}$

Then if $u=$ the firſt Year's Rent, due without Intereſt.

$\left.\begin{array}{l} Ru = \text{the Intereſt} \\ 2u = \text{the Rent} \end{array}\right\}$ due at the End of the ſecond Year.

$\left.\begin{array}{l} 2Ru = \text{the Intereſt} \\ 3u = \text{the Rent} \end{array}\right\}$ due at the End of the third Year.

$\left.\begin{array}{l} 3Ru = \text{the Intereſt} \\ 4u = \text{the Rent} \end{array}\right\}$ due at the End of the fourth Year.

$\left.\begin{array}{l} 4Ru = \text{the Intereſt} \\ 5u = \text{the Rent} \end{array}\right\}$ due at the End of the fifth Year.

And ſo on for any Number of Years. Hence it is evident, that $Ru+2Ru+3Ru+4Ru+5u=A$ the Sum of all the Rents and their Intereſt, being forborne 5 Years.

From

From whence it follows, that $Ru+2Ru+3Ru+4Ru=A-tu$. Here $t=5$. Divide by u, then $R+2R+3R+4R=\frac{A-tu}{u}$.

Next to find the Sum of this Progression (See Page 185) thus, Let $R+2R+3R+4R$ &c. $=z$, then $1+2+3+4$ &c. $=\frac{z}{R}$. Here the Sum of the first and last Terms are $4+1=5=t$, and the Numbers of all the Terms is $4=t-1$. Therefore $\frac{t-1}{2}\times t=$ the Sum of all the Terms; that is, $\frac{tt-t}{2}=\frac{z}{R}$: hence $\frac{ttR-tR}{2}=z$. Consequently $\frac{ttR-tR}{2}=\frac{A-tu}{u}$. Now from this Equation it will be easy to deduce the following *Theorems*.

Theorem 1. $\frac{ttRu-tRu+2tu}{2}=A$, or $\frac{ttu-u}{2}R:+tu=A$.

Theorem 2. $\frac{2A}{ttR-tR+2t}=u$. *Theorem* 3. $\frac{2A-2tu}{ttu-tu}=R$.

Let $-1=x$, then $t=\sqrt{\frac{2A}{Ru}+\frac{xx}{4}}:-\frac{1}{2}x$ *Theorem* 4.

Question 1. *If* 250 l. *yearly Rent (or Pension,* &c.*) be forborn or unpaid seven Years; what will it amount to in that Time, at* 6 per Cent. *for each Payment, as it becomes due?*

Here is given $u=250$, $t=7$, and $R=0{,}06$; to find A. *Per. Th.* 1. First $250\times7=1750=tu$, $1750\times7=12250=ttu$. Again $12250-1750=10500=ttu-tu$, and $\frac{10500}{2}\times0{,}06=315$. Lastly $315+1750-=2065=A$; *Viz.* 2065 *l.* is the *Answ.* required.

But if the Annuity, Rent or Pension, is to be paid by Quarterly or half Yearly Payments, &c. Then $\frac{0{,}06}{2}=0{,}03=R$ for half yearly Payments: and $\frac{0{,}06}{4}=0{,}015=R$ for quarterly: or $0{,}045=R$ for three quarterly Payments. Example of half yearly Payments.

Suppose 250 l. per Annum, *to be paid by half yearly Payments, were in Arrears, or unpaid for seven Years; what would it amount to, allowing* 6 per Cent. per Annum *for each Payment, as it becomes due?*

In this Example there is given $u = 125 = \frac{250}{2}$; $t = 14$ the Number of Payments; and $R = 0{,}03 = \frac{0{,}06}{2}$: thence to find A.

First $125 \times 14 = 1750 = tu$; $1750 \times 14 = 24500 = ttu$: again $24500 - 1750 = 22750 = ttu - tu$; then $\frac{22750}{2} = 11375$, and $11375 \times 0{,}03 = 341{,}25$. Laftly $341{,}25 + 1750 = 2091{,}25$; that is, $A = 2091$ *l.* 5 *s.* the Anfwer required.

N. B. Hence it may be obferved, that half yearly Payments are more advantageous than yearly. For 2091 *l.* 5 *s.* > 2065 *l.* by 26 *l.* 5 *s.* confequently, quarterly Payments are more advantageous than half yearly Payments.

Queftion 2. *What yearly Rent, Penfion, &c. being forborn or unpaid feven Years, will raife a Stock of* 2065 l. *allowing* 6 per Cent. per Annum *for each Payment, as it becomes due?*

Here is given $A = 2065$, $t = 7$, and $R = 0{,}06$; to find u. *Per Theorem* 2. First $7 \times 0{,}06 = 0{,}42 = tR$, and $0{,}42 \times 7 = 2{,}94 = ttR$. Then $ttR - tR = 2{,}52$. Laftly $ttR - tR + 2t = 16{,}52)\ 4130 = 2A\ 250 = u$; that is, 250 *l. per Annum,* &c. will raife 2065 *l.* the Stock required.

Queftion 3. *In what time will* 250 l. *yearly Rent raife a Stock of* 2065 l. *allowing* 6 per Cent, *&c. for the Forbearance of the Payments as they become due?*

Here is given $u = 250$. $A = 2065$, and $R = 0{,}06$; to find t. *Per Theorem* 4. First $\frac{2}{R} = \frac{2}{,06} = 33{,}3333$; and $33{,}3333 - 1 = 32{,}3333 = x = \frac{2}{R} - 1$; Then $16{,}16666$ &c. $= \frac{1}{2}x$; $261{,}3605$ &c. $= \frac{1}{4}xx$. Again $\frac{4130}{15} = 275{,}3333 = 2A \div Ru$, and $275{,}3333 + 261{,}3605 = 536{,}6938 = \frac{2A}{Ru} + \frac{1}{4}xx$. Then $\sqrt{536{,}6938} = 23{,}1666$. Laftly, $23{,}1666 - 16{,}1666 = 7 = t$ the Time required.

Queftion 4. *If* 250 l. *yearly Rent, being forborn feven Years, will amount to* 2065 l. *allowing Simple Intereft for every Payment as it becomes due; what muft the Rate of the Intereft be* per Cent. *&c.*

Here is given $u = 250$, $A = 2065$, and $t = 7$; to find R: *Per Theorem* 3.

$$\text{Thus} \begin{cases} ttu = 12250 \\ tu = 1750 \end{cases} \qquad \begin{cases} 4130 = 2A \\ 3500 = 2tu \end{cases}$$

$$ttu - tu = 10500)\quad 630 = 2A - 2tu\ (0{,}06 = R.$$

Then $1 : 0{,}06 :: 100 : 6$ the Rate required.

Sect.

Sect. 3. *The* PRESENT *Worth of* ANNUITIES *or Penſions*, &c. *computed at Simple Intereſt.*

THE Buſineſs of purchaſing Annuities, or taking of Leaſes, &c. for any aſſigned Time, depends upon the true equating of the Principal or Money laid out on the Purchaſe, with the Annuity or Yearly Rent, by allowing (or diſcompting) the ſame Rate of Intereſt to both Parties. Which may be eaſily performed by duly applying the reſpective *Theorems* of the two laſt Sections together; as will fully appear by the following Queſtion.

Queſtion 1. *What is* 75 l. *yearly Rent, to continue nine Years, worth in ready Money, at* 6 per Cent. per Annum *Simple Intereſt?*

1. *Per Theorem* 1. of the laſt Section, find what the propoſed yearly Rent would amount to, if it were forborn 9 Years, at 6 *per Cent.*

Thus $u=75$, $t=9$, and R 0,06: Quære A.

$$ttu=6075 \qquad \text{Then } 2)\,5400\,(2700 \Big\} \text{Multiply}$$
$$tu=\ 675 \qquad R=0{,}06$$

$$ttu-tu=5400 \qquad \left.\begin{array}{r}162,\\ +tu=675,\end{array}\right\}=837=A.$$

2. Then by *Theorem* 2. *Section* 1. find what Principal, being put to Intereſt for the ſame Time, and at the ſame Rate, will amount to 837 $l.=A$. Thus $tR=0{,}54=9\times 0{,}06$; $tR+1=1{,}54)$ 837 $(543{,}5064=P$: that is, $P=543\,l.$ 10 *s.* $1\frac{1}{2}d.$ which is the Worth of 75 *l.* a Year, as was required.

From the Work of theſe two Operations (duly conſidered) it muſt needs be eaſy to conceive, how the two *Theorems* by which they were performed, may be combined in one.

For 1. $\dfrac{ttRu-tRu+2tu}{2}=A$; and 2. $PtR+P=A$.

Conſequently $PtR+P=\dfrac{ttRu-tRu+2tu}{2}$. And from this Equation may be deduced the following *Theorems.*

Theorem 1. $\dfrac{ttRu-tRu+2\,2tu}{2tR+2}=P$, or $\dfrac{ttR-tR+2t}{2tR+2}\times a=P.$

By this *Theorem* all Queſtions of the ſame Kind with the laſt (*viz.* that above) may be eaſily and readily anſwered at one Operation.

Theorem 2. $\frac{2PtR+2P}{ttR-tR+2t}=u$, or $\frac{tR+1}{ttR-tR+2t}: \times 2P=u$.

Theorem 3. $\frac{2P-2tu}{ttu-tu-2Pt}=R$.

Let $-\frac{2P}{u}-1=x$, then will $tt+xt=\frac{2P}{Ru}$. Which gives this *Theorem* 4. $\sqrt{\frac{2P}{Ru}+\frac{xx}{u}}: +\frac{x}{2}=t$.

By the second and fourth *Theorems*, two very useful Questions may be easily answered.

1. *As for instance: If it be required to find what Annuity, or yearly Rent,* &c. *may be purchased, for any proposed Sum, to continue any assigned Time, allowing any Rate of Interest?*

This Question may be answered by *Theorem* 2.

2. *Again: If it be required to find how long any yearly Rent, Pension, or Annuity,* &c. *may be purchased (or enjoyed) for any proposed Sum, at any given Rate of Interest?*

All Questions of this Kind are easily answered by *Theorem* 4.

In these Questions it is supposed, that the Purchase or yearly Rent, is to commence or be immediately entered upon, But if it be required to find the Value or Purchase of an Annuity or yearly Rent, &c. in Reversion; that is, when it is not to be entered upon until after some Time, or Number of Years are past; then you must first find what the Sum proposed to be laid out in the Purchase, would amount to, if it were put to Interest, during the time the Annuity, &c. is not to be put in present Possession; and make that Amount the Sum for the Purchase, proceeding with it as in either of the two last Questions, &c.

Note, *From the first Question of this Section it will be easy to conceive how to perform the Equation of Payments, between Debtor or Creditor, at any Rate of Interest, without doing any Damage to either Party.*

That is, when several Sums of Money are to be paid, at several different Times, to find the Time when all the Payments may be truly discharged at once; as if one Sum were to be paid at the End of two Months, another at six Months, and perhaps a third Sum at eight Months End, &c. And if it were required to find the Time when all those Sums may be truly discharged at one Payment without Loss, &c.

CHAP.

CHAP. XII.

Of COMPOUND *Intereſt, and* ANNUITIES, *&c.*

COMPOUND Intereſt is that which ariſes from any Principal and its Intereſt put together, as the Intereſt ſo becomes due; ſo that at every Payment, or at the Time when the Payments became due, there is created a new Principal; and for that Reaſon it is called Intereſt upon Intereſt, or Compound Intereſt.

As for Inſtance; Suppoſe 100*l.* were lent out for two Years, at 6 *per Cent. per Annum*, Compound Intereſt: then at the End of the firſt Year, it will only amount to 106*l.* as in Simple Intereſt. But for the ſecond Year this 106 *l.* becomes Principal, which will amount to 112*l.* 7*s.* $2\frac{1}{2}$*d.* at the ſecond Year's End, whereas by Simple Intereſt it would have amounted to but 112*l.*

And although it be not lawful to let out Money at Compound Intereſt; yet in purchaſing of Annuities or Penſions, &c. and taking Leaſes in Reverſion, it is very uſual to allow Compound Intereſt to the Purchaſer for his ready Money; and therefore it is very requiſite to underſtand it.

Sect. I. *Of* COMPOUND *Intereſt.*

$$\text{Let}\begin{cases} P = \text{the Principal put to Intereſt.} \\ t = \text{the Time of its Continuance.} \\ A = \text{the Amount of the Principal and Intereſt.} \\ R = \begin{cases}\text{the Amount of 1 } l. \text{ and its Intereſt for 1 Year, at} \\ \text{any given Rate, which may be thus found.}\end{cases}\end{cases} \text{as before (for } P, t, A\text{).}$$

Viz. 100 : 106 :: 1 : 1,06 = the Amount of 1 *l.* at 6 *per Cent.*
Or 105 : 100 :: 1 : 1,05 = the Amount of 1 *l.* at 5 *per Cent.*
and ſo on for any other aſſigned Rate or Intereſt.

Then if R = the Amount of 1 *l.* for one Year, at any Rate.
R^2 = the Amount of 1 *l.* for two Years.
R^3 = the Amount of 1 *l.* for three Years.
R^4 = the Amount of 1 *l.* for four Years.
R^5 = the Amount of 1 *l.* for five Years. Here $t = 5$

For $1 : R :: R : RR :: RR : RRR :: RRR : R^4 :: R^4 : R^5$: &c. in $\div\!\!\div$.

That is $\begin{cases}\text{As one Pound : is to the Amount of one Pound at one} \\ \text{Year's End :: ſo is that Amount : to the Amount of} \\ \text{one Pound at two Year's End, \&c.}\end{cases}$

Whence

Whence it is plain, that Compound Intereſt is grounded upon a Series of Terms, increaſing in Geometrical Proportion continued; wherein t (*viz.* the Number of Years does always aſſign the Index of the laſt and higheſt Term: *Viz.* the Power of R, which is R^t.

Again, As $1 : R^t :: P : PR^t = A$ the Amount of P for the Time, that $R^t =$ the Amount of 1 *l.*

That is { *As one Pound : is to the Amount of one Pound for any given Time :: ſo is any propoſed Principal (or Sum) to it's Amount for the ſame Time.*

From the Premiſes (I preſume) the Reaſon of the following *Theorems* may be very eaſily underſtood.

Theorem 1. $PR^t = A$, as above.

From hence the two following *Theorems* are eaſily deduced.

Theorem 2. $\frac{A}{R^t} = P$. *Theorem* 3. $\frac{A}{P} = R^t$.

By theſe three *Theorems*, all Queſtions about Compound Intereſt may be truly reſolved by the Pen only, *viz.* without Tables; though not ſo readily as by the Help of Tables, calculated on Purpoſe; as will appear farther on.

Queſtion 1. *What will* 256l. 10s. *amount to in ſeven Years, at* 6 per Cent. per Annum. *Compound Intereſt?*

Here is given $P = 256,5$; $t = 7$; and $R = 1,06$, which being involved until its Index $= t$ (*viz.* 7.) will become $R^t = 1,50363$. Then $1,50363 \times 256,5 = 385,6811 = A = 385$ *l.* 13 *s.* $7\frac{1}{2}$ *d.* which is the Anſwer required.

Queſtion 2. *What Principal or Sum of Money muſt be put (or lent) out to raiſe a Stock of* 385 l. 13 s. $7\frac{1}{2}$d. *in ſeven Years, at* 6 per Cent. per Annum *Compound Intereſt?*

Here is given $A = 385,6811$; $R = 1,06$; and $t = 7$; to find P. by *Theorem* 2. Thus $R^t = 1,50363$) $385,6811 = A$($256,5 = P$. That is, $P = 256$ *l.* 10 *s.* which is the Principal or Sum, as was required.

Queſtion

Queſtion 3. *In what Time will* 256 l. 10s. *raiſe a Stock of (or amount to)* 385 l. 13s. 7½d, *allowing* 6 per Cent. per Annum *Compound Intereſt?*

Here is given $P = 256{,}5$; $A = 385{,}6811$; $R = 1{,}06$; to find t by the third *Theorem* $R^t = \frac{A}{P} = \frac{385{,}6811}{256{,}5} = 1{,}50363$, which being continually divided by $R = 1{,}06$ until nothing remain, the Number of thoſe Diviſions will be $7 = t$. Thus 1,06) 1,50363 (1,41852. And 1,06) 1,41852 (1,332225. Again 1,06) 1,338225 (1,262477. And ſo on until it become 1,06) 1,06 (1. which will be at the ſeventh Diviſion. Therefore it will be t 7 the Numbers of Years required by the Queſtion.

Queſtion 4. *If* 256 l. 10s. *will amount to (or raiſe a Stock of)* 385 l. 13s. 7½d. *in ſeven Years Time; what muſt the Rate of Intereſt be,* per Cent. per Annum?

Here is given $P=256{,}5$; $A=385{,}6811$, and $t=7$, Quære R. By *Theorem* 3. $\frac{A}{P} = R^t = 1{,}50363$; as before in the laſt Queſtion. And if $R^t = R^7 = 1{,}50363$, then $R = \sqrt[7]{1{,}50363}$, which may be thus extracted.

Put	1	$r+e=R$, then
1 ⊙ 7	2	$r^7+7r^6e+21r^5ee=R^7=1{,}50363=G$
2—r^7	3	$7r^6e+21r^5ee=G-r^7$
$3 \div 7r^5$	4	$re+3ee=\frac{G-r^7}{7r^5}=D$
$4 \div$	5	$e=\frac{D}{r+3e}$; let $r=1$, then $D=0{,}0719$

Operation $r = 1{,}00$
$+3e = 0{,}18$

Diviſor 1,18) 0,0719 (1,00 = r; 0,06 = e
708
1,06 $= r+e = R$
11 to be rejected.

Then 1 : 0,06 :: 100 : 6 the Rate *per Cent.* required.

The firſt three Queſtions may be much more eaſily performed by the following Table, which is only the Amount of one Pound for thirty-nine Years.

That

That is, of *R* . *RR* . *RRR* . R^4 . R^5 . and ſo on to R^{39}.

Years = *t*	The Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.	Years = *t*	The Amounts of 1 *l.* at 6 *per Cent*, &c. Compound Intereſt	Years = *t*	The Amounts of 1 *l.* at 6 *per Cent.* &c. Compound Intereſt.
1	1.06 = *R*	14	2.2609039557	27	4.8223459407
2	1.1236 = *R R*	15	2.3965582931	28	5.1116866971
3	1.191016 = *R*3	16	2,5403516847	29	5.4183878990
4	1.23247696	17	2.6927727857	30	5.7434911729
5	1.3382255776	18	2.8543391529	31	6.0881006432
6	1.4185191122	19	3,0255995021	32	6.4533866818
7	1.5036302590	20	3.2071353722	33	6.8405898828
8	1.5938480745	21	3.3995636005	34	7.2510252757
9	1.6894782590	22	3.6035374166	35	7.6860867923
10	1.7908476965	23	3.8197496616	36	8.1472519998
11	1.8982985583	24	4.0489346413	37	8.6360872198
12	2.0121964718	25	4.2918707197	38	9.1542523470
13	4.1329282601	26	4.5493829629	39	9.7035074878

The Title of this Table ſhews it's Conſtruction, and it's Uſe will eaſily appear by an Example or two.

EXAMPLE 1.

What will 375 l. 10s. *amount to in nine Years, at* 6 per Cent. per Annum?

The tabular Number againſt 9 Years is 1,689479, which being multiplied with the Principal 375,5 will produce 634,3993 &c. *viz.* 634 *l.* 8 *s. ferè*, being the Amount or Anſwer required.

EXAMPLE 2.

What Principal (or Sum) muſt be put to Intereſt to raiſe a Stock of 634 *l.* 8 *s in nine Years Time, at* 6 per Cent. per Annum?

If the propoſed Stock (*viz.* 634,4) be divided by the tabular Number that is againſt the given Number of Years (*viz.* 9.) the Quotient will be the Principal (or Sum) required. *Viz.* againſt 9 is 1.689479. Then 1.689479) 634,4 (375,5 = 375 *l.* 10 *s.* the Principal (or Sum) required.

EXAMPLE 3.

In what Time will 375 *l.* 10 *s. raiſe a Stock of (or amount to)* 634 *l.* 8 *s. at* 6 per Cent.

Divide

Divide the propoſed Stock (*viz.* 634,4) by the given Principal (*viz.* 375,5) and the Quotient will ſhew the tabular Number that ſtands over againſt the Time ſought. Thus 375,5) 634,4 (1,689479 *&c.* this Number being ſought in the Table, will be found to ſtand againſt 9 Years which is the Time required.

But if the Quotient cannot be truly found in the Table of Amounts for Years, as above; then take out of that Table the neareſt Number that is leſs, and make it a Diviſor, by which you muſt divide the firſt Quotient; and then ſeek the ſecond Quotient in the Table of Amounts for Days (which is inſerted a little further on) and it will aſſign the Number of Days: as in this Example.

In what Time will 563 l. *amount to* 860 l. *at* 6 per Cent. per Annum, *Compound Intereſt?*

Anſwer. In 7 Years and 99 Days.

Thus 563) 860 (1,52753 which ſhews the Time to be more (or above) ſeven Years; for over againſt 7 Years is 1,50363 which being made the new Diviſor: *Viz.* 1,50363) 1,52753 (1,01589) *&c.* this Number is the neareſt Amount to 99 Days.

Note, *If the Stock, Principal, and Time be given; the Rate of Intereſt will be beſt found by extracting the Root,* &c. *as before in the fourth Queſtion.*

The next Thing that I ſhall here propoſe, is to make this Table (which is only calculated for the Rate of 6 *per Cent.*) univerſally uſeful for all the Rates of Compound Intereſt, *which I may preſume to ſay, is a new Improvement of my own,* being well ſatisfied it never was publiſhed before; and not only ſo, but I have heard ſeveral good Artiſts affirm it was impoſſible to be done.

The Method of performing it is briefly thus, Let $x =$ the Difference between $1,06 = R$ the Amount of 1 *l.* for one Year (in the Table) and any other propoſed Amount of 1 *l.* for one Year; which admits of two Caſes.

Caſe 1. If the propoſed Rate be greater than the $1,06 = R$, then will $R + x =$ the true Amount of 1 *l.* for one Year at that Rate.

Caſe 2. But if the propoſed Rate be leſs than $1,06 = R$, then it will be $R - x =$ the Amount of 1 *l.* &c.

Make $\begin{cases} t-1=b,\ t-2=c,\ \ t-3=d,\ \ t-4=f, \text{ \&c. } \frac{1}{2}tb \\ =g,\ \frac{1}{3}cg=m,\ \frac{1}{4}dm=n,\ \frac{1}{5}fn=s \text{ \&c.} \end{cases}$

 Then

Then will $R^t + t R^b x + g R^c x^2 + m R^d x^3$ &c. = the Amount of 1 *l.* at the given Rate, for any Time denoted by t, in Case 1. And $R^t - t R^b x + g R^c x^2 - m R^d x^3$ &c. = the Amount of 1 *l.* in Case 2.

Which is no more but this: Let $R + x$ or $R - x$ (which soever it is) be involved (as directed in *Sect.* V. *Chap.* II.) to the same Power or Height as the Index t the given Time in the Question denotes: rejecting all the Powers of x above xxx or $xxxx$ at most, as useless. Then multiply that Power of $R + x$ or $R - x$ into the given Principal, and their Product will be the Amount required.

An Example or two in each Case will render all easy.

EXAMPLE 1.

Suppose it were required to find what 256 l. *would amount to in fifteen Years, at* 8 l. per Cent per Annum *Compound Interest?* Here $t = 15$.

First 100 : 108 : : 1 : 1,08 the Amount of 1 *l.* at 8 *per Cent.* Next 1,08, — 1,06 = 0,02 = x. And $R + x$. 1,08 as in Case 1. Then $R^{15} + 15\, R^{14} x + 105\, R^{13} xx + 455\, R^{12} xxx$ &c. = the Amount of 1 *l.* for 15 Years, at 8 *per Cent.*

Here $x = 0{,}02$. $xx = 0{,}0004$. and xxx ,000008

$$\text{By the Table } R^{15} = 2{,}396558$$

$$\text{And} \begin{cases} 15\, R^{14}\, x = 2{,}260904 \times 15 \times ,02 = 0{,}678271 \\ 105\, R^{13}\, xx = 2{,}132928 \times 105 \times ,0004 = 0{,}089583 \\ 455\, R^{12}\, xxx = 2{,}012196 \times 455 \times ,000008 = 0{,}007324 \end{cases}$$

$$\text{Sum} = 3{,}171736$$

Then $3{,}171736 \times 256 = 811{,}964416 = A$, That is, 811 *l.* 9 *s.* 3½*d. ferè.* Which is the Answer required.

EXAMPLE 2.

What will 365 l. *amount to in seven Years at four and a half* per Cent, *&c.*

First 100 : 104,5 : : 1 : 1,045 the Amount of 1 *l.* at 4½ *l. per Cent.*

Next 1,06—1,045=0,015=x. Consequently $R - x = 1{,}045$ as in Case 2.

Then $R^7 - 7\, R^6 x + 21\, R^5 xx - 35\, R^4 xxx$ &c. = the Amount of 1 *l.* for 7 Years, at 4½ *per Cent.*

Here

Here $x = ,015$; $xx = ,000225$; and $xxx = 000003375$

$$\begin{array}{lrl} & \text{By the Table } R^7 = + & 1,503630 \\ \text{And} \left\{ \begin{array}{l} -7\,R^6\,x \\ +21\,R^5\,xx \\ -35\,R^4\,xxx \end{array} \right. & \begin{array}{r} = - \\ = + \\ = - \end{array} & \begin{array}{l} 0,148944 \\ 0,006323 \\ 0,000141 \end{array} \end{array}$$

$$R^7 - 7\,R^6\,x + 21\,R^5\,xx - 35\,R^4\,xxx = 1,360868$$

Then $1,360868 \times 365 = 496,71682 = A$.

That is, 496 *l.* 14 *s.* 3 $\frac{1}{4}$*d.* is the Answer required.

If the Reason of these two Operations be but well understood, it will be very easy to conceive how to find P, the Principal, by having A, t, and x given (because R and it's Powers are always given by the Table).

For $\overline{R^t \pm t\,R^b\,x + g\,R^c\,xx \pm m\,R^d\,xxx} \times P = A$ (as above.)

Therefore $\dfrac{A}{R^t + t\,R^b\,x + g\,R^c\,xx + m\,R^d\,xxx} = P$.

Or if A, P, and t, be given, x may be found.

For $R^t \pm t\,R^b\,x + g\,R^c\,xx \pm m\,R^d\,xxx = \dfrac{A}{P}$. This Equation being solved (as in Chap. X.) the Value of x will be found; and then either $R + x$, or $R - x$ will shew the Rate of Interest, &c.

But I shall leave the numerical Operations to the Learner's Practice, supposing enough done to shew how all Questions of this Kind that are limited by whole Years may be computed.

And if the Time given or sought be not terminated by whole Years, but by Weeks, Months, Quarters, or Half-Years, &c. for resolving such Questions, the best Way will be to reduce those Parts of a Year into Days; that done, find an Answer according to the Demand of the Question (and agreeing to one 1 *l.* as before) for that Number of Days; and in order to that, it will be requisite to find the Amount of 1 *l.* for one Day *(as in my Compendium of Algebra,* Page 110) which I shall here insert.

Put $a =$ the Amount sought, then it will be

$$1 : a :: a : aa :: aa : aaa :: aaa : aaaa \;\therefore\; \text{to } a^{365}.$$

That is $\left\{\begin{array}{l}\textit{As one Pound is to its Amount for one Day :: so is that} \\ \textit{Amount : to the Amount of two Days :: and so is that} \\ \textit{of two Days : to that of three Days, And so on in} \div \textit{to} \\ \textit{365 Days.}\end{array}\right.$

 Then

Then the last of the Terms will be $a^{365} = 1,06$

Put	1	$r + e = a$. And let $r = 1$
$1 \odot^{365}$	2	$r^{365} + 365\,r^{364}\,e + 66430\,r^{363}\,ee = a^{365} = 1,06$
2 in Numb.		$1 + 365\,e + 66430\,ee = 1,06$
—	3	$365\,e + 66430\,ee = 0,06$
$3 - 1$	4	$,00549\,e + ee = 0,0000009032 = D$
$4 \div 66430$	5	
$5 \div$	6	$e = \dfrac{D}{,00549 + e}$

Operation ,00549
$+\ e =$,0001

1st Divisor ,00559) 0,0000009032 . . . . . $\begin{cases} 1,0000000 = r \\ 0,0001598 = e \end{cases}$
$+\ e =$,00015 — 559 — $1,0001598 = r + e = a$
2d Divisor ,00574 —)3442 — true to the 7th Figure
$+\ e =$,000059 — 2870 — and only too much by
3d Divisor ,005799 —)57200 — 2 in the 8th, at one
&c. — &c. — Operation.

Now $r =$ 1,00016 for the second Operation. Then

2 in Numb.	7	$1,06013401407 + 386,887\,e + 70402,172\,ee = 1,06$. Hence it appears that $r - e = a$.
Therefore	8	$1,06013401407 - 386,887\,e + 70402,172\,ee = 1,06$
$8 +$	9	$386,887\,e - 70402,172\,ee = 0,00013401407$
$9 \div$	10	$,0054953 - ee = ,000000000190355\,03$
$10 \div$	11	$e = \dfrac{,00000000019035503}{,0054953 - e}$

Operation, ,0054953
$-\ e =$ 3

1st Divisor ,0054950) 0,00000000019035503 . . . . $\begin{cases} 1,00016 = r \\ 0,000000346417 = e \end{cases}$
34 — 164850 — $1,000159653583 = r - e$
,00549466 —)2550503
$-\ e =$ 46 — 2197864
2d Divisor ,005494614 —)35263900
$-\ e =$ 64 — 32967684
3d Divisor, ,00549460 —)2296216
2197840
98376
54956
&c.

Which being further pursued to a third Operation will give $a = 1,00015965358745\,3$ &c.

This

This Value of *a* is the Amount of 1 *l.* for one Day, from which, if 1 *l.* be ſubſtracted, the Remainder = ,000159653587 &c. will be the Intereſt of 1 *l.* for one Day. Conſequently, if any propoſed Principal be multiplied into either of theſe, the reſpective Product will be the Amount or Intereſt of that Principal for one Day, at 6 *per Cent.* &c.

And that the Amount (or Intereſt) of any Principal or Sum may be eaſily computed for any Number of Days leſs than a Year; I have here inſerted the following Table, which with a great deal of Care (and I believe Exactneſs) is calculated from the laſt found (1,000159653587453) Amount of 1 *l.* for one Day. To which alſo is annexed a Table of the Amounts of 1 *l.* for Months.

Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.
1	1.0001596536	26	1.0041592879	51	1.0081749166
2	1.0003193326	27	1.0043196055	52	1.0083358753
3	1.0004790372	28	1.0044799487	53	1.0084968597
4	1.0006387673	29	1.0046403175	54	1.0086578699
5	1.0007985229	30	1.0084007120	55	1.0088189057
6	1.0009583039	31	1.0049611320	56	1.0089799673
7	1.0011181105	32	1.0051215776	57	1.0091410545
8	1.0012779426	33	1.0052820488	58	1.0093021675
9	1.0014378002	34	1.0054425457	59	1.0094633062
10	1.0015976834	35	1.0056030682	60	1.0096244707
11	1.0017575920	36	1.0057636164	61	1.0097856608
12	1.0019175262	37	1.0059241901	62	1.0099468767
13	1.0020774859	38	1.0060847895	63	1.0101081184
14	1.0022374712	39	1.0062454146	64	1.0102693858
15	1.0023974820	40	1.0064060653	65	1.0104306789
16	1.0025575184	41	1.0065667416	66	1.0105919978
17	1.0027175803	42	1.0067274436	67	1.0107533424
18	1.0028776677	43	1.0078881712	68	1.0109147128
19	1.0030377808	44	1.0070489245	69	1.0110761090
20	1.0031979193	45	1.0072097035	70	1.0112375309
21	1.0033580850	46	1.0073705082	71	1.0113989786
22	1.0035182732	47	1.0075313385	72	1.0115604521
23	1.0036784885	48	1.0076921945	73	1.0117219513
24	1.0038387294	49	1.0078530762	74	1.0118834764
25	1.0039989958	50	1.0080139835	75	1.0120450272

Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.	Days	Amounts of 1 *l.* &c.
76	1.0122066038	116	1.0186908655	156	1.0252166658
77	1.0123682062	117	1.0188535031	157	1.0253803453
78	1.0125298344	118	1.0190161667	158	1.0255440509
79	1.0126914885	119	1.0191788563	159	1.0257077827
80	1.0128531683	120	1.0193415719	160	1.0258715406
81	1.0130148739	121	1.0195043134	161	1.0260353247
82	1.0131766054	122	1.0196670809	162	1.0263629713
83	1.0133383627	123	1.0198298745	163	1.0261991349
84	1.0135001458	124	1.0199926934	164	1.0265268338
85	1.0136619547	125	1.0201555389	165	1.0266907225
86	1.0138237895	126	1.0203184110	166	1.0268546374
87	1.0139856501	127	1.0204813084	167	1.0270185784
88	1.0141475365	128	1.0206442319	168	1.0271825456
89	1.0143094488	129	1.0208071814	169	1.0273465389
90	1.0144713869	130	1.0209701569	170	1.0275105585
91	1.0146333511	131	1. 211331585	171	1.0276746046
92	1.0147953408	132	1.0212961861	172	1.0278386764
93	1.0149573565	133	1.0214592397	173	1.0280027746
94	1.0151193981	134	1.0216223193	174	1.0281668989
95	1.0152814655	135	1.0217854250	175	1.0283310494
96	1.0154435589	136	1.0219485567	176	1.0284952262
97	1.0156056781	137	1.0221117144	177	1.0286594291
98	1.0157678232	138	1.0222748982	178	1.0288236583
99	1.0159299941	139	1.0224381081	179	1.0289879137
100	1.0160921910	140	1.0226013440	180	1.0291521953
101	1.0162544138	141	1.0227646060	181	1.0293160231
102	1.0164166624	142	1.0229278940	182	1.0294908372
103	1.0165789370	143	1.0230902081	183	1.0296451975
104	1.0167412375	144	1.0232545483	184	1.0298095841
105	1.0169035638	145	1.0234179146	185	1.0299739969
106	1.0170659161	146	1.0235813069	186	1.0301384359
107	1.0172282944	147	1.0237447253	187	1.0303029012
108	1.0173906985	148	1.0239081699	188	1.0304673928
109	1.0175513286	149	1.0240716405	189	1.0306319206
110	1.0177155846	150	1.0242351372	190	1.0307964557
111	1.0178780665	151	1.0243986600	191	10.309610251
112	1.0180405744	152	1.0245622089	192	10.311256216
113	1.0182031083	153	1.0247257830	193	1.0312902445
114	1.0183656680	154	1.0248893851	194	1.0314548937
115	1.0185282578	155	1.0250530124	195	1.0316195692

Days

Days	Amounts of 1*l.* &c.	Days	Amounts of 1*l.* &c.	Days	Amounts of 1*l.* &c.
196	1.0317842709	236	1.0383939484	276	1.0450459680
197	1.0319489990	237	1.0385597318	277	1.0452128133
198	1.0321137534	238	1.0387255415	278	1.0453796853
199	1.0322785341	239	1.0388913778	279	1.0455446584
200	1.0324433410	240	1.0390572405	280	1.0457135092
201	1.0326081742	241	1.0392231298	281	1.0458804611
202	1.0327730339	242	1.0393890454	282	1.0460474397
203	1.0329379198	243	1.0395549876	283	1.0462144449
204	1.0331028321	244	1.0397209563	284	1.0463814768
205	1.0332677706	245	1.0398869515	285	1.0465484353
206	1.0334327355	246	1,0400529732	286	1.0467156200
207	1.0335977268	247	1.0402190214	287	1.0468827325
208	1.0337627444	248	1.0403850961	288	1.0470498711
209	1.0339277883	249	1.0405511973	289	1.0472170363
210	1.0340928586	250	1.0407173250	290	1.0473842283
211	1.0342579552	251	1.0408834793	291	1.0475514469
212	1.0344230782	252	1.0410496601	292	1.0477186923
213	1.0345882275	253	1.0412158674	293	1 0478859643
214	1.0347534033	254	1.0413821012	294	1.0480532631
215	1.0349186054	255	1.0415483616	295	1.0482205885
216	1.0350838338	256	1.0417146485	296	1.0483879407
217	1.0352490887	257	1.0413809620	297	1.0485553196
218	1.0354143699	258	1.0420473021	298	1.0487227252
219	1 0355796775	259	1.0422136687	299	1.0488901576
220	1.0357450115	260	1.0423800618	300	1.0490576166
221	1.0359103719	261	1.0425464815	301	1.0492251025
222	1.0360757587	262	1.0427129278	302	1.0493926150
223	1.0362411719	263	1.0428794007	303	1.0495601543
224	1.0364066116	264	1.0430459001	304	1.0497277204
225	1.0365720776	265	1.0432124261	305	1.0498953132
226	1.0367375701	266	1.0433789787	306	1.0500629327
227	1.0369030889	267	1.0435455579	307	1.0502305790
228	1.0370686342	268	1.0437121637	308	1.0503982521
229	1.0372342059	269	1.0438787961	309	1.0505659519
230	1.0373998041	270	1.0440454551	310	1.0507336786
231	1.0375654287	271	1.0442121407	311	1.0509014320
232	1.0377310798	272	1.0443788529	312	1.0510692121
233	1.0378967573	273	1.0445455918	313	1.0512370191
234	1.0380624612	274	1.0447123572	314	1.0514048529
235	1.0382241916	275	1.0448791493	315	1.0515727134

Days	Amounts of 1*l.* &c.	Days	Amounts of 1*l.* &c.	Days	Amounts of 1*l.* &c.
316	1.0517406008	339	1.0556094165	362	1.0594924636
317	1.0519085150	340	1.0557779484	363	1.0596616154
318	1.0520764559	341	1.0559465071	364	1.0598307942
319	1.0522444237	342	1.0561150927	365	1.06
320	1.0524124183	343	1.0562837053		
321	1.0525804397	344	1.0564523448		
322	1.0527484880	345	1.0566210112		
323	1.0529165631	346	1.0567897045		
324	1.0530846650	347	1.0569584248		
325	1.0532527937	348	1.0571271720		
326	1.0534209493	349	1.0572959594		
327	1.0535891317	350	1.0574647472		
328	1.0537573410	351	1.0576335753		
329	1.0539255771	352	1.0578024303		
330	1.0540938401	353	1.0579713122		
331	1.0542621300	354	1.0581402211		
332	1.0544304467	355	1.0583091570		
333	1.0545987903	356	1.0584781199		
334	1.0547671608	357	1.0586471097		
335	1.0549355582	358	1.0588161265		
336	1.0551039824	359	1.0589851703		
337	1.0552724336	360	1.0591542411		
338	1.0554409116	361	1.0593233389		

Months	The Amounts of 1*l.* at 6 *per Cent.* For Months.
1	1.0048675505
2	1.0097587942
3	1.0146738462
4	1.0196128224
5	1.0245758394
6	1.0295630141
7	1.0345744641
8	1.0396103076
9	1.0446706634
10	1.0497556507
11	1.0548653894
12	1.06

The Uſe of this Table is in all reſpects like that of whole Years, in finding the Amount of any given Sum for any propoſed Number of Days leſs than a Year.

EXAMPLE 1.

Suppoſe it were required to find the Amount of 375 l. for 210 *Days at* 6 per Cent.

The Amount of 1*l.* for 210 Days is 1,0340928 *&c.* per Table. Then 1,0340928 × 375 = 387,7848 *&c.* = 387 *l.* 15 *s.* $8\frac{1}{4}$*d.* which is the Amount required. And the reſt of the Variations may be performed juſt as in the Examples of whole Years.

But if the Time given conſiſts of Years, and Parts of a Year; as Quarters, Months, *&c.* Then reduce the odd Time or Parts of the Year into Days; and the Anſwer may then be found at two Operations; as in the following *Example.*

EXAMPLE

Example 2. *Suppoſe it were required to find what* 265 *l.* *would amount to in five Years and* 135 *Days at* 6 per Cent. *&c.*

First the Amount of 1 *l.* for $\begin{cases} 5 \text{ Years is } 1{,}338225 \text{ &c.} \\ 135 \text{ Days is } 1{,}021785 \text{ &c.} \end{cases}$

Then $1{,}338225 \times 1{,}021785 \times 265\,l. = 362{,}355232$, &c, being the *Amount* or *Anſwer* required.

Or, if the *Amount* and *Time* are given, to find the *Principal*; Then *Multiply* the *Amount* of 1 *l.* for the *Years*, and the *Amount* of 1 *l.* for the odd Days together; And by their *Product* divide the given *Amount*, the *Quotient* will be the *Principal* required.

Example 3. *What Principal will raiſe a Stock of* 362 *l* 7 *s.* $1\frac{1}{2}$*d.* *Or* 362,355232 *l.* in 5 *Years and* 135 *Days, at* 6 per Cent. *&c.*

The Amount of 1 for $\begin{cases} 5 \text{ Years is } 1{,}338225 \text{ &c.} \\ 135 \text{ Days is } 1{,}021785 \text{ &c.} \end{cases}$

Then $1{,}338225 \times 1{,}021785 = 1{,}367378$ &c. the *Diviſor*. Next 1,367378) 362,355232 = *A* (265 *l.* the *Principal* required.

Again, if the *Principal* and its *Amount* are given, to find the *Time*, at 6 *per Cent*, &c. you muſt divide the *Amount* by its *Principal*, and then proceed as in the *Third Example*, Page 256, for the *Anſwer required.*

But if the *Amount* and its *Principal*, with the *Time* of its being at *Intereſt*, are given, to find the *Rate* of *Intereſt*; Then proceed as in the Fourth *Queſtion*, *Page* 255, &c.

Now in order to make this *Table* of *Amounts* for Days, uſeful for all *Rates* of *Intereſt* (*as before in that for Years*) you muſt firſt find the Simple *Intereſt* of 1 l. for one Day, both at the given Rate, and alſo at 6 *per Cent*. And call their Difference *x*.

Thus, ſuppoſe the given *Ratio* were 8 per Cent. per Annum, Firſt 100 : 8 : : 1 : 0,08 And 100 : 6 : : 1 : 0,06 the Two Simple *Intereſts* for one Year.

Then 365) 0,08 (0,00021917 &c. the *Simple Intereſt* of 1 l. for one Day, at 8 per Cent.

And 365) 0,06 (0,00016438 &c. the *Simple Intereſt* of 1 l. for one Day, at 6 per Cent.

Their *Difference* 0,00005479 = *x* which may do indifferently well for ordinary ſmall *Queſtions*; but where Exactneſs is required, it will be convenient to make Uſe of this *Proportion*.

Viz. { As the Simple Intereſt of 1 *l.* for one Day at 6 *per Cent*: Is to the Tabular Intereſt of 1 *l.* for one Day :: So is the Simple Intereſt of 1 *l.* for one Day, at any given Rate: To a Fourth Number.

That is, 0,00016438 : 0,00015965 :: 0,00021917 : 0,00021286
Then $0,00021286 - 0,00015965 = 0,00005321 = x$.

This x being *involved* with the reſpective *Amounts* for Days, in the ſame Manner as was done with thoſe for *Years* (*vide Page* 258) the Reſult will be the *Anſwer* to the *Queſtion*.

Sect. 2. ANNUITIES or PENSIONS *in Arrear, computed at Compound Intereſt.*

WHEN *Annuities*, &c. are ſaid to be in *Arrear*, ſee Page 248. And I ſhall here make uſe of the ſame Letters to repreſent the ſame things as before in that Page, ſave only that R is here equal to the *Amount* of 1 *l.* as in *Section* 1. of this *Chapter*.

Suppoſe $u =$ the Firſt *Year's Rent* of any Annuity without *Intereſt*.

Then will $Ru + u =$ { the *Amount* of the Firſt *Year's Rent*, and its *Intereſts*; More the 2d *Year's Rent*.

And $RRu + Ru + u =$ { the *Amount* of the 1ſt and 2d *Years Rents*, with their *Intereſts*; More the 3d *Year's Rent*, &c.

Here $RRu + Ru + u = A$ the *Amount* of any *Yearly Rent* or *Annuity*, being forborn *Three Years*. And from hence may be deduced theſe *Proportions*.
Viz. $u : Ru :: Ru : RRu :: RRu : RRRu$ and ſo on in $\div\!\!\div$ for any *Number* of *Terms* or *Years* denoted by t, wherein the laſt Term will always be uR^{t-1}.
Conſequently $A - uR^{t-1} =$ the Sum of all the Antecedents
And $A - u =$ the Sum of all the Conſequents in the Series.
And therefore it would be $u : uR :: A - uR^{t-1} : A - u$ Vide *Page* 188.
Ergo $Au - uu = RuA - uuR^{t}$ which, being divided all by u, will become $A - u = RA - uR^{t}$.

From this laſt *Equation* it will be eaſy to raiſe the following *Theorems*.

Theorem 1. $\left\{\dfrac{uR^{t} - u}{R - 1} = A.\right.$ *Theorem* 2. $\left\{\dfrac{RA - A}{R - 1} = u.\right.$

Theorem

Theorem 3. $\left\{\frac{RA+u-A}{u}\right.=R^t$. If this *Equation* be continually *divided* by R, until nothing *remain*, the *Number* of thoſe *Diviſions* will be t. See Page 255.

Theorem 4. $\left\{\frac{A}{u}R-R^t=\frac{A-u}{u}\right.$. If this *Equation* be reſolved into *Numbers*, according to the Method propoſed in *Sect.* 3. *Chap.* 10. the Root will ſhew the Value of R.

QUESTION 1. *If* 30 *l. Yearly Rent, or Annuity,* &c. *be forborn* (i. e. *remain unpaid) Nine Years; what will it amount to, at* 6 per Cent per Annum, *Compound Intereſt?*
Here is given $u=30$, $t=9$, and $R=1{,}06$; to find A. per *Theorem* 1.

$$\begin{array}{l} R^9=1{,}689479 \text{ By the Table of } \textit{Amounts} \text{ for } \textit{Years} \\ \qquad\qquad 30=u \\ \hline R^9u=50{,}684370 \\ -u=30, \\ \hline \end{array}$$

$R-1=0{,}06)$ $20{,}684370$ $(344{,}7395=344\,l.\ 14\,s.\ 9\tfrac{1}{2}d.=A$ the Amount required,

QUESTION 2. *What Yearly Rent or Annuity,* &c. *being forborn or unpaid Nine Years, will raiſe a Stock of* 344 *l.* 14 *s.* $9\tfrac{1}{2}$*d.* $=344{,}7395$, *at* 6 per Cent. &c.
Here is given $A=344{,}7395$, $t=9$, and $R=1{,}06$; to find u, per *Theorem* 2.

$$\begin{array}{r} AR=344{,}7395\times1{,}06=365{,}42387 \\ -A=344{,}7395 \\ \hline \end{array}$$

$R^t-1=1{,}689479-1=0{,}682479)$ $20{,}68437$ $(30=u.$

QUESTION 3. *In what Time will* 30 *l. Yearly Rent raiſe a Stock or Amount to* 344 *l.* 14 *s.* $9\tfrac{1}{2}$*d. allowing* 6 per Cent. *for the Forbearance of Payment?*

Here is given $u=30$, $A=344{,}7395$, and $R=1{,}06$; to find t. per *Theorem* 3.

Firſt $AR+u-A=365{,}42387+30-344{,}7395=50{,}68437$. And $u=30)$ $50{,}68437$ $(1{,}689479=R^t$. Then $R=1{,}06)$ $1{,}689479$ $(1{,}593848$. And $1{,}06)$ $1{,}593848$ $(1{,}50363$; and ſo on until it become $1{,}06)$ $1{,}06$ $(1.$ which will be at the *Ninth Diviſion*; therefore $t=9$.

Or $R^t = 1{,}689479$, being sought in the *Table* of *Amounts* for *Years*, will be found to stand over-against 9 Years, which is the *Time required.*

QUESTION 4. *If* 30*l.* per Annum, *being unpaid Nine Years, will amount to* 344*l.* 14*s.* $9\frac{1}{2}$*d. allowing Compound Interest for every Payment as it becomes due, What must the Rate of Interest be* per Cent. *&c.*

Here is given $u=30$, $A=344{,}7395$, and $t=9$; to find R by the last of the Four *Equations*, *Viz.* $\left\{ \frac{A}{u}R - R^t = \frac{A-u}{u} \right.$

First $\frac{A}{u} = \frac{344{,}7395}{30} = 11{,}491317$. And $\frac{A-u}{u} = 10{,}491317$.

Hence there is this *Equation*; $11{,}491317\ R - R^9 = 10{,}491317$

Let	1	$r+e=R$, and suppose $r=1$
1 ⊚ 9	2	$r^9 + 9r^8e + 36\,r^7\,ee = R^9$
1 in Numb.	3	$11{,}491317 + 11{,}491317\,e = 11{,}491317\,R$
2 in Numb.	4	$1{,}000000 + 9{,}000000\,e + 36\,ee = R^9$
3—4	5	$10{,}491317 + 2{,}491317\,e - 36\,ee = 10{,}491317$
Whence	6	$36\,ee = 2{,}491317\,e$
6 ÷ 36 e	7	$e = 0{,}06$ &c.

$\left.\begin{array}{l} \text{First } r = 1 \\ + e = 0{,}06 \end{array}\right\} = 1{,}06 = R$ $\left\{\right.$ *As may be easily try'd by involving it, and ordering it, as the Equation above directs.*

Section 3. *To find the* PRESENT WORTH *of Annuities, Pensions, or Leases,* &c. *at Compound Interest.*

LET $P=$ the *present Worth* of any *Annuity*, or *Lease*, &c. and the rest of the Letters as before.

Then, from what has been said in *Section* 3. *Chap.* 11. about *Purchasing* of *Annuities*, &c. at Simple Interest, it will be easy to form the like *Theorems* here at *Compound Interest*, *viz.* by Combining *Theorem* 1. *Page* 266. and *Theorem* 1. *Page* 254 into one *Theorem*.

For $\left\{ \frac{uR^t - u}{R-1} = A \right.$ $\left\{\right.$ *The Amount of any Yearly Rent being unpaid any Number of Years. Per* Theorem 1. *of the last Section.* Page 266.

And $PR^t = A$ $\left\{\right.$ *The Amount of any Principal or Sum being put to Interest, for the same Number of Years. Per* Theorem 1. *Page* 254.

Hence

Hence is follows, That $PR^t = \frac{uR^t - u}{R - 1}$,

Viz. $PR^{t+1} - PR^t = uR^t - u$ being the very ſame *Equation* with that in my *Compendium* of *Algebra*, Page 112. which is there raiſed from the Conſideration of purchaſing *Annuities*, or taking of *Leaſes*, &c. to be grounded upon a *Rank* or *Series* of *Geometrical Proportionals* continually decreaſing. Thus $\frac{u}{R}$ is the *Firſt* and *Greateſt Term*; R the common *Ratio* of all the *Terms*; and P is the *Sum* of all the *Series*.

That is, $\frac{u}{R} : \frac{u}{RR} :: \frac{u}{RR} : \frac{u}{RRR} :: \frac{u}{RRR} : \frac{u}{R^4} :: \frac{u}{R} : \frac{u}{R^5}$ &c.

in $\div\!\div$ until the laſt *Term* $= \frac{u}{R^t}$. Then will $P - \frac{u}{R^t}$ be the *Sum* of all the *Antecedents*, and $P - \frac{u}{R}$ the *Sum* of all the *Conſequents*.

Therefore it will be

$\frac{u}{R} : \frac{u}{RR}$. Or (in the ſame Ratio) $u : \frac{u}{R} :: P - \frac{u}{R^t} : P - \frac{u}{R}$

which produces $PR^{t+1} - uR^t = PR^t - u$. As above.

From this *Equation* may be deduced the following *Theorems*.

Theorem 1. $\left\{ \frac{u - \frac{u}{R^t}}{R - 1} = P. \right.$ *Theorem* 2. $\frac{\overline{PR^t \times R} : - PR^t}{R^t - 1} = u.$

Theorem 3. $\left\{ \frac{u}{P + u - PR} = R^t \right.$ $\left\{\right.$ *Which, being continually divided by* R, *will give* t.

Theorem 4. $\left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^{t+1} \right.$. The reſolving of which *Equation* will diſcover the Value of R.

Queſtion 1. *What is* 30 *l. Yearly Rent, to continue Seven Years, worth in ready Money, allowing* 6 per Cent. *Compound Intereſt to the Purchaſer?*

Here is given $u = 30$. $t = 7$. And $R = 1{,}06$ to find P. per *Theorem* 1. *Viz.* $\frac{u}{R^t} = \frac{30}{1{,}50363} = 19{,}9517$.

And $30 - 19{,}9517 = 10{,}0483 = u - \frac{u}{R^t}$.

Then

Then $R-1=0{,}06)\ 10{,}0483\ (167{,}4716=P=167$ l. 9s. 5d. being the Answer required.

Question 2. *What Annuity or Yearly Rent, to continue Seven Years, may be purchased for* 167 l. 9s. 5d. allowing 6 per Cent. *Compound Interest to the Purchaser?*

In this *Question* there is given $P=167{,}4716$. $t=7$
And $R=1{,}06$ to find u. By the Second *Theorem*.
First $PR^t\times R=251{,}8153\times 1{,}06=266{,}9242$
And $-PR^t=167{,}4716\times 1{,}50363)=251{,}8153$

Then $R^t-1=0{,}50363)\quad 15{,}1089\quad (30=u$
That is $u=30\,l.$ the Answer required.

Question 3. *How long may one have a Lease of* 30 *l. Yearly Rent, for* 167 l. 9s. 5d. *allowing* 6 per Cent. *Compound Interest to the Purchaser?*

Here is given $P=167{,}4716$. $u=30$. And $R=1{,}06$ to find t. By the Third *Theorem*.
First $P+u=167{,}4716+30=197{,}4716$
And $-PR=177{,}5199$

Then $19{,}9517)\,30=u(1{,}50363=R^t$

If this $1{,}50363=R^t$ be either continually *divided* by $1{,}06=R$ until nothing remain (as before in Page 255.) Or if it be sought in the Table of Amounts for Years, &c. it will discover $t=7$ which is the true *Answer* required.

Question 4. *Suppose one should give* 167 l. 9s. 5d. *for the Purchase of a Pension, or Annuity of* 30 l. per Annum, *to continue Seven Years; At what Rate of Interest*, per Cent. *would that Purchase be made, allowing Compound Interest to the Purchaser?*

In this Question there is given, $P=167{,}4716$. $u=30$ and $t=7$ to find R. Per *Theorem* 4 in this *Equation* $\left\{+\frac{u}{P}=\frac{u}{P}\right.$ $R^t+R^t-R^{t+1}$ which being brought into Numbers, and its *Root extracted*, as in the fourth Question of the last *Section*; the Value of R will be found 1,06, and then it will be 1 : 0,06 :: 100 : 6 the Rate per Cent. as was required.

These

Theſe four Queſtions include all the Varieties that can be propoſed about purchaſing Annuitiies or Leaſes, &c. which are to be either immediately entered upon, or in Poſſeſſion at the Time when the Purchaſe is made.

But ſuch Queſtions as relate to Annuities, or a taking of Leaſes, &c. in Reverſion, muſt be parted or divided into two diſtinct Queſtions, each to be ſeparately conſidered by itſelf (ſee Page 252.) As in the following *Examples*.

Example 1. " Suppoſe it were required to compute the preſent " Worth of 75 *l.* Yearly Rent, which is not to commence or be " entered upon, until Ten Years hence; and then to continue 7 " Years after that Time: at 6 *per Cent.* &c. Compound Intereſt?"

The Firſt Work in this *Queſtion* is, to find what 75 *l.* per Annum, to continue ſeven Years, is worth in ready Money; as if it were to be immediately entered upon: And to perform that, there is given $u=75$. $R=1{,}06$. and $t=7$. to find P. as in the firſt Queſtion of this *Section*.

Thus, $\frac{u}{R^t}=\frac{75}{1{,}50363}=49{,}8793$. And $75-49{,}8793=25{,}1207$

$=u-\frac{u}{R}$

Then, $R-1=0{,}06)\ 25{,}1207=418{,}6783=$ 418 l. 14 s. $6\frac{1}{4}$ d. the *Anſwer* to the firſt Part of the Queſtion.

Then the next Work will be, to find what Principal or Sum being put out ten *Years*, at 6 per Cent. &c. will amount to 418 l. 14 s. $6\frac{1}{4}$d. Here is given $A=418{,}6783$, $R=1{,}06$, $t=10$. to find P. per *Theorem* 2. Page 254.

Thus $R^{10}=1{,}790847)\ 418{,}6783=A\ (233{,}7884=$ 233 l. 15 s. 9 d. the preſent Worth of 75 l. per Annum in Reverſion, &c. As was required.

Example 2. " What Annuity or Yearly Rent to be entered " upon Ten Years hence, and this to continue Seven Years, may " be purchaſed for 233 l. 15 s. 9 d. ready Money, at 6 per Cent. " &c. Compound Intereſt?

In the 1ſt Work of this Queſtion there is given, $P=233{,}7884$ $R=1{,}06$. And $t=10$ (the Time which the Annuity is not to be entered upon) to find A. per *Theorem* 1. Page 254.

Thus, $PR^t=233{,}7884\times 1{,}790847=418{,}6783=A$ the Amount

Amount of 233 l. 15 s. 9 d. put to Intereſt Ten Years, at 6 per Cent. &c. Then for the Second Work of the Queſtion there is given $P=418{,}6783$. $R=1{,}06$. And $t=7$ (the Time that the Annuity is to be enjoyed) to find u. Per *Theorem* 2. of this *Section*.

$$\text{Thus } PR^t \times R = 418{,}6783 \times 1{,}50363 \times 1{,}06 = 667{,}3095$$
$$-PR^t = 418{,}6783 \times 1{,}50363 = 629{,}5372$$

$$R^t - 1 = 0{,}50363)\ \ 37{,}7723\ (\ 75 = u$$

That is, $u=75$ l. the Yearly Rent required by the *Queſtion*.

Theſe two Examples of finding P and u do fully ſhew the Method that muſt be uſed in Reſolving the two general, and indeed, the moſt uſeful Queſtions about Annuities or Leaſes in Reverſion: And if there be Occaſion, either the *Rate*, or the *Time*, viz. R or t, may be found by a due Application of their reſpective *Theorems*.

Note, "That which hath been done in the two laſt Sections "about Annuities or Yearly Rents, &c. at 6 per Cent, may alſo "be done for any Rate of Intereſt, by applying the Difference of "the Rates (viz. x.) As directed in the firſt Section of this Chap- "ter."

Now becauſe that Rents and Annuities, &c. are uſually paid either by Quarterly or Half Yearly Payments, and the Method of computing them by the Pen may be thought a little troubleſome; I have inſerted the following Tables of the Amounts of 1 l. for each, at 6 per Cent.

Half Years $= t$.	Annuities of 1 *l.* at 6 *per Cent.* Compound Intereſt.	Half Years $= t$.	Annuities of 1 *l.* at 6 *per Cent.* Compound Intereſt.	Half Years $= t$.	Annuities of 1 *l.* at 6 *per Cent.* Compound Intereſt.
1	1,0295630141	11	1,3777875592	21	1,8437905523
2	1,06	12	1.4185191122	22	1,8982985583
3	1,0913367949	13	1,4604548127	23	1,9544179853
4	1,1236	14	1,5036302590	24	2,0121964718
5	1,1568170026	15	1,5480821017	25	2,0716830644
6	1,191016	16	1,5938480745	26	2,1329282601
7	1,2262260228	17	1,6409670276	27	2,1959840483
8	1,26247696	18	1,6894789589	28	2,2609039557
9	1,2997995842	19	1,7394250493	29	2,3277430912
10	1,3382255776	20	1,7908476965	30	2,3965581931

Quar-

Quarterly Amounts.

Quarters of a Year = t.	Amounts of 1*l.* at 6 *per Cent.* &c. Compound Intereſt.	Quarters of a Year = t	Amounts of 1*l.* at 6 *per Cent.* &c. Compound Intereſt.	Quarters of a Year = t.	Amounts of 1*l.* at 6 *per Cent.* &c. Compound Intereſt.
1	1,0146738461	21	1,3578624938	41	1,8171263199
2	1,0295630141	22	1,3777875592	42	1,8437905523
3	1,0446706634	23	1,3980050019	43	1,8708460509
4	1,06	24	1,4185191122	44	1,8982985583
5	1,0755542769	25	1,4393342435	45	1,9261538989
6	1,0913367949	26	1,4604548127	46	1,9544179853
7	1,1073509032	27	1,4818853020	47	1,9830968140
8	1,1236	28	1,5036302590	48	2,0101964718
9	1,1400875335	29	1,5256942978	49	2,0417231330
10	1,1568170026	30	1,5480821017	50	2,0716830644
11	1,1737919574	31	1,5707984203	51	2,1020826228
12	1,191016	32	1,5938480745	52	2,1329282601
13	1,2084927856	33	1,6172359557	53	2,1642265211
14	1,2262260228	34	1,6409670276	54	2,1959840483
15	1,2442194748	35	1,6950463253	55	2,2282075801
16	1,26247696	36	1,6894789589	56	2,2609039557
17	1,2810023527	37	1,7142701133	57	2,2940801123
18	1,2997995842	38	1,7394250493	58	2,3277430912
19	1,3188726433	39	1,7649491048	59	2,3619000349
20	1,3582255776	40	1,7908276965	60	2,3965581931

Either of theſe *Tables* may alſo be made uſeful for any propoſed *Rate* of *Intereſt*; by making the $\frac{1}{2}$ or $\frac{1}{4}$ of the *Difference* of the *Rate* $= z$, &c.

As for Inſtance, ſuppoſe any of the aforeſaid *Queſtions* about *Annuities* or *Rents*, &c. were to be computed at 8 *per Cent. per Annum.*

Then 1,08—1,06=0,02=x for *Yearly Payments*: as before.
Conſequently 2) 0,02 (0,01=x for *Half Year's Payments.*
Or 4) 0,02 (0,005=x for *Quarterly Payments.*

Now theſe Values of x, although they are not really true, yet they may ſerve indifferently well for ſmall *Rents*; as I have already ſaid, *Page* 265. But if you would work exactly;

Then $\sqrt{}$ 1,08=1,0392304845 &c.
— $\sqrt{}$ 1,06=1,0295630141 *Vide Table, Page* 272.

Difference = 0,0096674704 = x for $\frac{1}{2}$ *Yearly Payments.*

And $\sqrt{}:\sqrt{}\ 1{,}08=1{,}0194263092$ &c.
$-\sqrt{}:\sqrt{}\ 1{,}06=1{,}0146738461$ See the *Last Table*.

Their Difference $0{,}0047524631=x$. for *Quarterly Payments*.

These are the true *Values* of x, which being *involved* with their respective *Amounts* (as before for *Years*, &c.) according as the *Question* requires, the *Result* will be the *Answer* at 8 *per Cent.* &c. The like may be done for any other *Rate*, either *Greater* or *Less* than 6.

Now, although the Method used here (and in *Page* 257 and 258, &c.) be really true (by which the *Tables* calculated only for 6 *per Cent.* are made effectual for all Rates of *Compound Interest*) yet it was rather proposed to shew what may possibly be perform'd by the Pen without a great many *Tables* of several *Rates*, than intended for common Practice.

For it must needs be confess'd, that *Tables*, calculated on Purpose for any designed *Rate* of *Interest*, are much more ready and useful in common Practice. And therefore since the Legislative Power hath thought fit to reduce the *Rate* of *Interest*, and hath settled it by an Act of Parliament, at 5 *per Cent.* I've therefore been at the Trouble *(which was not a little)* to calculate the following *Tables* for that *Rate*; but don't think it convenient to take the *Tables* at 6 *per Cent.* out of the Book, because the Examples are all suited to them; and not only so, but they may be found useful in the taking of *Leases* for Houses, *&c.* For in those Cases, the *Purchaser* is allowed more *Interest* for his purchase Money than the common *Rate* paid upon the Loan of Money.

Here

Here follow New *Tables* of the *Amounts* of one Pound at the *Rate* of 5 *per Cent. per Annum* Compound Intereſt. For *Years*, *Half-Years*, *Quarters*, *Months*, and *Days*.

I. The *Table* of the *Yearly Amounts* of 1 *l.* &c.

Years = t	The Amounts of 1 *l.* &c.	Years = t	The Amounts of 1 *l.* &c.	Years = t	The Amounts of 1 *l.* &c.
1	1,05 = *R*	14	1,97993160	27	3,73345632
2	1,1025 = *R R*	15	2,07892818	28	3,92012914
3	1,157625 = R^3	16	2,18287459	29	4,11613599
4	1,21550625	17	2,29201832	30	4,32194239
5	1,27628156	18	2,40661923	31	4,53803919
6	1,34009564	19	2,52695019	32	4,76494147
7	1,40710042	20	2,65329770	33	5,00318854
8	1,47745544	21	2,78596259	34	5,25334797
9	1,55132822	22	2,92526072	35	5,51601536
10	1,62889463	23	3,07152375	36	5,79181613
11	1,71033936	24	3,22509994	37	6,08140694
12	1,79585633	25	3,38635494	38	6,38547729
13	1,88564914	26	3,55567269	39	6,70475115

II. The *Table* of the *Half-Yearly Amount* of 1 *l.* &c.

Half Yrs. = t	The Amounts of 1 *l.* &c.	Half Yrs. = t	The Amounts of 1 *l.* &c.	Half Yrs. = t	The Amounts of 1 *l.* &c.
1	1,02469507	11	1,30779943	21	1,66912030
2	1,05	12	1,34009564	22	1,71033936
3	1,07592983	13	1,37318940	23	1,75257632
4	1,1025	14	1,40710042	24	1,79585633
5	1,12972632	15	1,44184887	25	1,84020513
6	1,57625	16	1,47745544	26	1,88564914
7	1,18621264	17	1,51394132	27	1,93221539
8	1,21550625	18	1,55132822	28	1,97993160
9	1,24552327	19	1,58963838	29	2,02882616
10	1,27628156	20	1,62889463	30	2,07892818

III. The *Table* of the *Quarterly Amounts* of 1*l.* &c.

Quarters = t	The Amounts of 1*l.* &c.	Quarters = t	The Amounts of 1*l.* &c.	Quarters = t	The Amounts of 1*l.* &c.
1	1,01227223	21	1,29194439	41	1,64888480
2	1,02469507	22	1,30779943	42	1,66912031
3	1,03727037	23	1,32384905	43	1,68960414
4	1,05	24	1,34009564	44	1,71033936
5	1,06288585	25	1,35654161	45	1,73132904
6	1,07592983	26	1,37318940	46	1,75257632
7	1,08913389	27	1,39004151	47	1,77408435
8	1,1025	28	1,40710042	48	1,79585633
9	1,11603014	29	1,42436869	49	1,81789549
10	1,12972632	30	1,44184887	50	1,84020513
11	1,14359059	31	1,45954358	51	1,86278856
12	1,157625	32	1,47745544	52	1,88564914
13	1,17183164	33	1,49558712	53	1,90879027
14	1,18621264	34	1,51394132	54	1,93221539
15	1,20077012	35	1,53252076	55	1,95592799
16	1,21550625	36	1,55132822	56	1,97993160
17	1,23042323	37	1,57036648	57	2,00422978
18	1,24552327	38	1,58963838	58	2,02882616
19	1,26080862	39	1,60914680	59	2,05372439
20	1,27628156	40	1,62889463	60	2,07892818

IV. The *Table* of the *Monthly Amounts* of 1*l.* &c.

Months = t	The Amounts of 1*l.* &c.	Months = t	The Amounts of 1*l.* &c.	Months = t	The Amounts of 1*l.* &c.
1	1,00407412	5	1,02053728	9	1,03727037
2	1,00816485	6	1,02469507	10	1,04149634
3	1,01227223	7	1,02886981	11	1,04573953
4	1,01629636	8	1,03306155	12	1,05

NOTE: The *Amount* of one *Pound*, for one *Day*, is 1,0001336807225, *&c.* (found as that in *Page* 260) but in the following *Table*, I take only Nine of those *Figures*, as being sufficient in Practice, for computing the *Interest* of any *Sum* not exceeding One Hundred Millions of *Pounds*.

V. The

V. The *Table* of the *Daily Amounts* of 1 *l.* &c.

Days	The Amounts of 1*l.* &c.	Days	The Amounts of 1 *l.* &c.	Days	The Amounts of 1 *l.* &c.
1	1,00013368	36	1,00482376	71	1,00953587
2	1,00026738	37	1,00495810	72	1,00967082
3	1,00040109	38	1,00509245	73	1,00980579
4	1,00053483	39	1,00522681	74	1,00994079
5	1,00066158	40	1,00536119	75	1,01007579
6	1,00080235	41	1,0054955	76	1,01021083
7	1,00093614	42	1,00563000	77	1,01034587
8	1,00106994	43	1,00576443	78	1,01048093
9	1,00120377	44	1,00589888	79	1,01061602
10	1,00133761	45	1,00603335	80	1,01075112
11	1,00147147	46	1,00616784	81	1,01088623
12	1,00160535	47	1,00630234	82	1,01102137
13	1,00173924	48	1,00643987	83	1,01115652
14	1,00187315	49	1,00657141	84	1,01129169
15	1,00200708	50	1,00670597	85	1,01142688
16	1,00214103	51	1,00684055	86	1,01159209
17	1,00227500	52	1,00697514	87	1,01169732
18	1,00240899	53	1,00710975	88	1,01183256
19	1,00254299	54	1,00724438	89	1,01196783
20	1,00267701	55	1,00737903	90	1,01210311
21	1,00281105	56	1,00751370	91	1,01223841
22	1,00294510	57	1,00764839	92	1,01237372
23	1,00307918	58	1,00778309	93	1,01250906
24	1,00321327	59	1,00791781	94	1,01264441
25	1,00334738	60	1,00805255	95	1,01277978
26	1,00348151	61	1,00818731	96	1,01291517
27	1,00361565	62	1,00832208	97	1,01305058
28	1,00374982	63	1,00845687	98	1,01318600
29	1,00388400	64	1,00859168	99	1,01332145
30	1,00401820	65	1,00872651	100	1,01345691
31	1,00415242	66	1,00886136	101	1,01359239
32	1,00428665	67	1,00899623	102	1,01372788
33	1,00442091	68	1,00913111	103	1,01386340
34	1,00455518	69	1,00926601	104	1,01399893
35	1,00468947	70	1,00940091	105	1,01413448

Days = *t*.	The Amounts of 1*l.* &c.	Days = *t*.	The Amounts of 1*l.* &c.	Days = *t*.	The Amounts of 1*l.* &c.
106	1,01427005	146	1,01970775	186	1,02517459
107	1,01440564	147	1,01984406	187	1,02531164
108	1,01454125	148	1,01998039	188	1,02544870
109	1,01467687	149	1,02011675	189	1,02558578
110	1,01481252	150	1,02025312	190	1,02572288
111	1,01494818	151	1,02038950	191	1,02586000
112	1,01508386	152	1,02052591	192	1,02599714
113	1,01521955	153	1,02066234	193	1,02613430
114	1,01535527	154	1,02079878	194	1,02627147
115	1,01549100	155	1,02093524	195	1,02640866
116	1,01562675	156	1,02107170	196	1,02654588
117	1,01576252	157	1,02120822	197	1,02668310
118	1,01589831	158	1,02134473	198	1,02682015
119	1,01603412	159	1,02148127	199	1,02695762
120	1,01610994	160	1,02161782	200	1,02709490
121	1,01630578	161	1,02175439	201	1,02723221
122	1,01644164	162	1,02189098	202	1,02736953
123	1,01657752	163	1,02202758	203	1,02750686
124	1,01671349	164	1,02216421	204	1,02764422
125	1,01684933	165	1,02230085	205	1,02778160
126	1,01698527	166	1,02243751	206	1,02791899
127	1,01712122	167	1,02257419	207	1,02805640
128	1,01725719	168	1,02271089	208	1,02819384
129	1,01739317	169	1,02284761	209	1,02833129
130	1,01752918	170	1,02238494	210	1,02846875
131	1,01766521	171	1,02312109	211	1,02860624
132	1,01780125	172	1,02325787	212	1,02874375
133	1,01793731	173	1,02339466	213	1,02888127
134	1,01807338	174	1,02353147	214	1,02901881
135	1,01820948	175	1,02366829	215	1,02915637
136	1,01834559	176	1,02380514	216	1,02929395
137	1,01848173	177	1,02394200	217	1,02943154
138	1,01861788	178	1,02407888	218	1,02956916
139	1,01875405	179	1,02421575	229	1,02970679
140	1,01889024	180	1,02435270	220	1,02990445
141	1,01902644	181	1,02448964	221	1,02998212
142	1,01916267	182	1,02462659	222	1,03011980
143	1,01929891	183	1,02476350	223	1,03025751
144	1,01943517	184	1,02490055	224	1,03039524
145	1,01957145	185	1,02503756	225	1,03053289

Days

Days = t.	The Amount of 1l. &c.	Days = t.	The Amount of 1l. &c.	Days = t.	The Amount of 1l. &c.
226	1,03067074	266	1,03619636	306	1,04175160
227	1,03080852	267	1,03633488	307	1,04189086
228	1,03094632	268	1,03647342	308	1,04203015
229	1,03108414	269	1,03661197	309	1,04216944
230	1,03122197	270	1,03675055	310	1,04230876
231	1,03135983	271	1,03688914	311	1,04244810
232	1,03149770	272	1,03702775	312	1,04258245
233	1,03163559	273	1,03716638	313	1,04272683
234	1,03177350	274	1,03730503	314	1,04286624
235	1,03191143	275	1,03744370	315	1,04300563
236	1,03204938	276	1,03758239	316	1,04314506
237	1,03218734	277	1,03772109	317	1,04328451
238	1,03232533	278	1,03785982	318	1,04342397
239	1,03246333	279	1,03799856	319	1,04356346
240	1,03260135	280	1,03813732	320	1,04370297
241	1,03273939	281	1,03827609	321	1,04384249
242	1,03287744	282	1,03841489	322	1,04398203
243	1,03301552	283	1,03855371	323	1,04412159
244	1,03315361	284	1,03869254	324	1,04426117
245	1,03329173	285	1,03883139	325	1,04440077
246	1,03342986	286	1,03897027	326	1,04454038
247	1,03356801	287	1,03910916	327	1,04468002
248	1,03370617	288	1,03924817	328	1,04481967
249	1,03384436	289	1,03938699	329	1,04495934
250	1,03398157	290	1,03952594	330	1,04509903
251	1,03412079	291	1,03966491	331	1,04523874
252	1,03425903	292	1,03980389	332	1,04537847
253	1,03439729	293	1,03994289	333	1,04551822
254	1,03453557	294	1,04008191	334	1,04565798
255	1,03467387	295	1,04022095	335	1,04579777
256	1,03481218	296	1,04036001	336	1,04593757
257	1,03495052	297	1,04049908	337	1,04607739
258	1,03508887	298	1,04063818	338	1,04621723
259	1,03522724	299	1,04077729	339	1,04635709
260	1,03536563	300	1,04091642	340	1,04649697
261	1,03550404	301	1,04105557	341	1,04663686
262	1,03565247	302	1,04119474	342	1,04677678
263	1,03578091	303	1,04133393	343	1,04691671
264	1,03591938	304	1,04147314	344	1,04705[illegible]
265	1,03605786	305	4,04161236	345	1,04719661

Days = ?	The Amounts of 1 *l.* &c.	Days = ?	The Amounts of 1 *l.* &c.	Days = ?	The Amounts of 1 *l.* &c.
346	1,04733663	353	1,04831708	360	1,04929845
347	1,04747664	354	1,04845722	361	1,04943872
348	1,04761666	355	1,04859738	362	1,04957901
349	1,04775671	356	1,04873756	363	1,04971932
350	1,04789677	357	1,04887775	364	1,04985965
351	1,04803686	358	1,04901797	365	1,04999999
352	1,04817696	359	1,04915820	366	1,05

I think it needless to say any Thing of the Use of these *Tables* because I take it for granted, that whoever understands the Work of the foregoing *Examples*, at 6 *per Cent.* cannot but know how to make use of these *Tables* at 5 *per Cent.* as Occasion requires.

Thus far concerning *Annuities*, or *Leases*, &c. that are *limited* by any assigned *Time*; and 'tis only such that can be computed by *Theorems* or certain *Rules*. However it may not perhaps be *unacceptable*, to insert a brief Account of some *Estimates* that have been reasonably made, by *two* very ingenious *Persons*, about the Proportion or Difference of *Mens Lives*, according to their several *Ages*; which may be of good Use in computing the *Values* of *Annuities*, or taking of *Leases* for *Lives*, &c.

Sir *William Petty* in his Discourse made before the *Royal Society* (*Anno* 1674) concerning the Use of DUPLICATE PROPORTION, in the Life of *Man* and its *Duration*; saith, that it's found by *Experience* there are more Persons living of between 16 and 26 *Years Old*, than of another *Age* or *Decade* of Years in the whole Life of *Man* (*viz.* 70 or 80 *Years.*) *His Reason for that Assertion I shall omit*; but supposing it true, he thence infers, that the *Roots* of every *Number* of *Mens Ages* under 16 (whose *Root* is 4) compared with the said Number 4, doth shew the *Proportion* of the Likelihood of such Mens reaching the *Age* of 70 Years.

As for *Example*, 'tis 4 *Times* more likely, that One of 16 *Years Old* would live to 70, than a New-Born *Babe*: 'Tis 3 *Times* more likely, that One of 9 *Years Old* should attain the *Age* of 70, than the said *Infant*, &c.

On the other Hand, 'tis 5 to 4, that One of 25 *Years Old* will die before One of 16: And 6 to 5, that One of 36 will die before One of 25. And so on according to the *Roots* of any other declining *Age*, compared with (4, 6) the *Root* of 21, which is the *Year* of *Perfection* according to the Sense of our *Law*, and the *Age* for whose Life a *Lease* is most *valuable*.

2. The

2. The ingenious and great Mathematician, Doctor *Edmund Halley* (in *Philoſoph. Tranſact. Numb.* 196) doth, with great Induſtry and Skill, draw an Eſtimate of the Proportion of Mens Lives, from the *Monthly Tables* of the Births and Funerals in *Breſlaw*, the Capital City of the *Province* of *Sileſia*; or, as the *Germans* call it, *Schleſia.* Whence he proves that 'tis 80 to 1, a Perſon of 25 Years old will not die in a Year: That it is $5\frac{1}{2}$ to 1, that a Man of 40 will live 7 Years: That a Man of 30 Years Old may reaſonably expect to live 27 or 28 Years, *&c.*

Now from theſe and the like Proportions (he juſtly infers) that the Price of Inſurance upon Lives ought to be regulated, there being a great Difference between the Life of a Man of 20, and one of 50. For *Example:* 'Tis 100 to 1, that a Man of 20 dies not in a Year, and but 38 to 1, for a Man of 50 *Years* of Age. And upon theſe alſo depends the *Valuation* of Annuities for Lives: for it is plain, that the Purchaſer ought to pay only ſuch a Part of the Value of any Annuity, as he hath Chances that he is living.

And for that Purpoſe he hath taken the Pains *(which was not a little)* to compute the following Table (that ſhews the Value of *Annuities*) for every Fifth Year of Age to the 70th.

Age	*Year's Purch.*	*Age*	*Year's Purch.*	*Age*	*Year's Purch.*
1	10,28	25	12,27	50	9,21
5	13,40	30	11,72	55	8,51
10	13,44	35	11,12	60	7,60
15	13,33	40	10,57	65	6,54
20	12,78	45	9,91	70	5,32

The ſame ingenious *Gentleman* proceeds on, and ſhews how to eſtimate or find the Value of *Two Lives*, and then of *Three Lives*, which being too long a Diſcourſe to be recited here, I have, for Brevity's Sake, omitted it; and ſhall only add this ſerious Obſervation.

Viz. How unjuſtly we repine at the Shortneſs of our *Lives*, and think ourſelves wrong'd if we attain not to *Old Age*; whereas it appears, that the *One Half* of thoſe, that are BORN, *die* in Seventeen Years Time. For by the aforeſaid Bills of Mortality at *Breſlaw*, it was found, that 1238 were in that Time reduced to 616. So that, inſtead of murmuring at what we call a *ſhort Life*, we ought to account it as a great Bleſſing that we have *ſurviv'd* perhaps by many Years, that *Period* of *Life* whereat the one Half of the whole Race of Mankind does not arrive.

Sect. 4. *Of Purchasing* FREE-HOLD, *or* REAL ESTATES; *at Compound Interest.*

ALL *Free-hold* or *Real Estates*, are supposed to be purchased or bought to continue for ever (viz. *without any limited Time)*; therefore the Business of computing the true Value of such Estates is grounded upon a Rank or Series of *Geometrical Proportions* continually decreasing, *ad Infinitum.*

Thus, let P, u, R, denote the same *Data* as in the last *Section*. Then the *Series* will be, $\frac{u}{R}$, $\frac{u}{RR}$, $\frac{u}{R^3}$, $\frac{u}{R^4}$, $\frac{u}{R^5}$, and so on in $\div\!\!\div$ until the last *Term* $= 0$. Then will $P - 0$ (viz. P) be the Sum of all the *Antecedents*. And $P - \frac{u}{R}$ will be the *Sum* of all the *Consequents*; therefore it will be $u : \frac{u}{R} :: P : P - \frac{u}{R}$ which produces $PR - u = P$.

The *Equation* affords the following *Theorems*.

Theorem 1. $PR - P = u$. *Theorem* 2. $\left\{\frac{u}{R-1} = P.\right.$

Theorem 3. $\frac{P+u}{P} = R.$

Example. "Suppose a Free-hold Estate of 75 l. Yearly Rent "were to be sold; what is it worth, allowing the Buyer 6 *per Cent.* &c. Compound Interest for his Money?"

In this Question there is given $u = 75$. $R = 1{,}06$ to find P, Per *Theorem* 2. Thus $R - 1 = 0{,}06$) $75 = u$ (1250 l. $= P$. the *Answer* required. And so on for any of the rest, as Occasion requires. But if the *Rent* is to be paid, either by *Quarterly*, or *Half Yearly Payments*;

Then $R = \sqrt{\ }1{,}06$ for *Half Yearly* } *Payments at* 6 *per Cent.*
And $R = \sqrt{\ }:\sqrt{\ }1{,}06$ for *Quarterly* }

Or { $R = 1{,}08$ for *Yearly* }
{ $R = \sqrt{\ }1{,}08$ for *Half Yearly* } *Payments at* 8 *per Cent.*
{ $R = \sqrt{\ }:\sqrt{\ }1{,}08$ for *Quarterly* }

The like is to be understood for any other proposed *Rate* of *Interest*, either *greater* or *less* than 6 per Cent.

The Application of these *Theorems* to Practice is so very easy, that it is needless to insert more *Examples.*

AN INTRODUCTION TO THE MATHEMATICS.

PART III.

CHAP. I.

Of Geometrical Definitions, &c.

Sect. I. Of *Lines* and *Angles*.

A Point hath no Parts: That is, a *Geometrical Point* is not any *Quantity*, but only an *assignable Place* in any *Quantity*, denoted by a *Point*: As at *A*. and *B*. } *A*. *B*.

Such a Place may be conceived so infinitely small, as to be void of Length, Breadth, and Thickness; and therefore a Point may be said to have no *Parts*.

2. A Line is called a *Quantity* of one *Dimension*, because it may have any supposed *Length*, but no *Breadth* nor *Thickness*, being made or represented to the *Eye*, by the *Motion* of a *Point*.

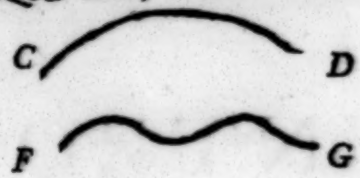
That is, if the *Point* at *A*, be moved (upon the same Plane) to the Point at *B*, it will describe a *Line* either right or circular (viz. crooked) according to its Motion.

Therefore the Ends or Limits of a Line are *Points*.

3. A Right Line, is that *Line* which lieth *even* or *streight* betwixt those Points that limit its *Length*, being the *shortest Line* that can be drawn between any *Two Points*. As the *Line A B*. } *A*————————*B*.

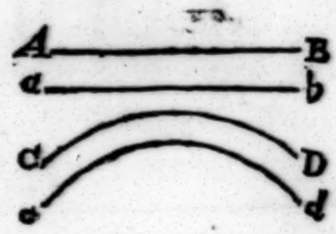
Therefore, between any two Points, there can lie or be drawn but one right Line.

4. A CIRCULAR, *crooked* or OBLIQUE *Line*, is that which lies bending between those *Points* which *limit* its *Length*, as the *Lines* *C D* or *F G*, &c.

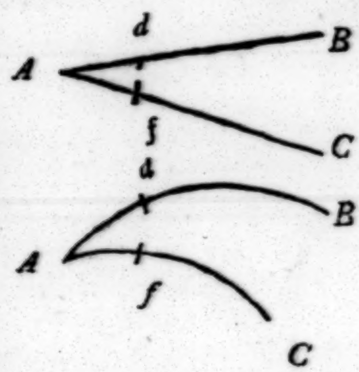


"Of these Kinds of Lines there are various Sorts; but those of the Circle, Parabola, Ellipsis, and Hyperbola are of most general Use in Geometry; of which a particular Account shall be given further on.

5. PARALLEL LINES, are those that lie *equally distant* from one another in all their *Parts*, viz. such *Lines* as being infinitely extended (upon the same Plane) will never meet: As the Lines *A B* and *a b*: or *C D* and *c d*.



6. LINES not PARALLEL, but INCLINING (*viz. leaning*) one towards another, whether they are *Right Lines*, or *Circular Lines*, will (if they are extended) meet, and make an *Angle*; the *Point* where they meet is called the *Angular Point*, as at *A*. And according as such *Lines* stand, nearer or further off each other, the *Angle* is said to be lesser or greater, whether the Lines that include the *Angle* be long or short. That is, the Lines *A d*, and *A f* include the same *Angle* as *A B* and *A C* do; notwithstanding that *A B* is longer than *A d*, &c.

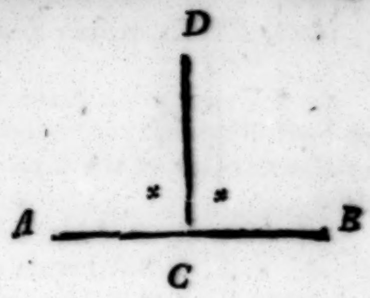


7. All ANGLES included between *Right Lines* are called *Right-lined Angles*; and those included between *Circular Lines* are called *Spherical Angles*. But all *Angles*, whether *Right-lined* or *Spherical*, fall under one of these *Three Denominations*.

Viz. { A RIGHT ANGLE.
An OBTUSE ANGLE.
An ACUTE ANGLE. }

8. A RIGHT-ANGLE is that which is *included* betwixt *Two Lines*, that *meet* one another *Perpendicularly*.

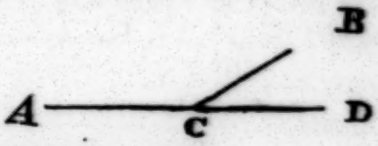
That



That is, when a *Right Line*, as *D C*, meets with another *Right-Line*, as *AB*, so directly as that it neither inclines nor declines to one Side more than the other, but makes the *Angles* on both Sides of it *equal*, as at *x x*; then are those *Angles* called *Right Angles*; and the *Lines* so meeting are said to be *Perpendicular* to each other.

That is, *AC*, and *CB*, are *Perpendicular* to *DC*, as well as *D, C* is to either or both of them.

9. An OBTUSE ANGLE is that which is greater than a *Right Angle*. Such is the *Angle* included between the *Lines AC* and *CB*.



10. An ACUTE ANGLE is that which is less than a *Right Angle:* As the *Angle* included between the *Lines CB and CD*.

These *Two Angles* are generally called OBLIQUE *Angles*.

Sect. 2. *Of a* CIRCLE, *&c.*

"BEFORE a Circle and its Parts are defined, it will be con-"venient to give a brief Account of Superficies in general."

1. A SUPERFICIES or SURFACE is the Upper, or very Out-side of any visible Thing. But by *Superficies* in GEOMETRY, is meant only so much of the Out-side of any Thing as is inclosed within a *Line* or *Lines*, according to the Form or Figure of the Thing designed; and it is produced or formed by the Motion of a *Line*, as a *Line* is described by the Motion of a *Point*; thus:

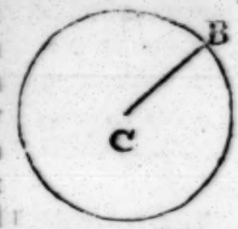
Suppose the *Line AB* were equally moved (upon the same Plane) to *CD*; then will the *Points* at *A* and *B* describe the two *Lines AC* and *BD*; and by so doing they will form (and inclose) the SUPERFICIES or *Figure ABCD*, being a *Quantity of Two Dimensions*, viz. it hath *Length* and *Breadth*, but not *Thickness*. *Consequently the Bounds or Limits of a Superficies are Lines.*

Note,

Note, *The Superficies of any Figure, is usually called its* AREA.

2. A CIRCLE is a *plain regular Figure*, whose *Area* is *bounded* or *limited* by one continued *Line*, called the CIRCUMFERENCE or PERIPHERY of the *Circle*, which may be thus described or drawn.

Suppose a *Right Line*, as *CB*, to have one of its *Extream Points*, as *C*, so fixed upon any *Plane*, as that the other *Point* at *B* may move about it; then if the *Point* at *B* be moved round about (upon the same *Plane*) it will describe a *Line* equally distant in all its Parts from the *Point C*, which will be the *Circumference* or *Periphery* of that *Circle*; the *Point C*, will be its CENTER, and the contained Space will be its *Area*, *and the Right Line C B, by which the Circle is thus described, is called* RADIUS.



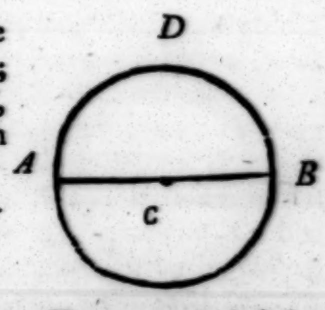
Consectary.

" From hence it is evident, that an infinite Number of Right " *Lines* may be drawn from the Center of any Circle to " touch its Periphery, which will be all equal to one another, be- " cause they are all *Radius*'s. And with a little Consideration it " will be easy to conceive, that no more than two equal Right " Lines can be drawn from any Point within a Circle to touch " its Periphery, but from the Center only." (9.)

3. EQUAL CIRCLES are those which have equal *Radius*'s; for it is plain by the last *Definition*, that one and the same *Radius* (at *CB*) must needs describe equal *Circles*, how many soever they be.

4. The *Diameter* of a *Circle*, is twice its *Radius* joined into one Right *Line*; as *A B* drawn through the Center *C*, and ending at the *Periphery* on each *Side*.

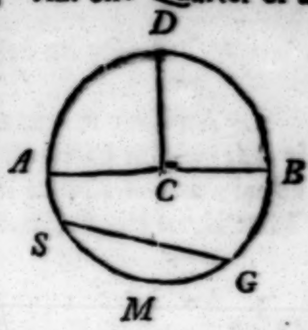
That is, the *Diameter* divides the *Circle* into Two equal Parts.



5. A *Semicircle* (viz. Half a *Circle*) is a Figure included between the *Diameter*, and Half the *Periphery* cut off by the *Diameter*; as *A D B*.

6. A

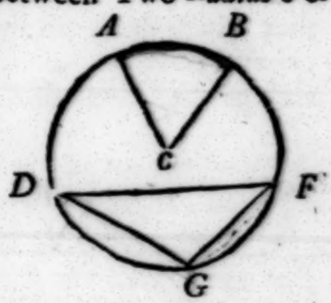
6. A QUADRANT is half a Semicircle, viz. one Quarter of a *Circle*; and it is made by the Radius (as *DC*) standing Perpendicular upon the Diameter of the Center *C*, cutting the *Periphery* of the Semicircle in the Middle, as at *D*. " Therefore a Quadrant, or " half the Semicircle, is the Measure of " a Right Angle."



7. A CHORD LINE, or the Subtense of an Arch, is any *Right Line* that cuts the *Circle* into two unequal Parts, as the Line *SG*; and is always less than the Diameter.

8. A SEGMENT of a *Circle*, is a Figure included betwixt the *Chord* and that Arch of the *Periphery* which is cut off by the *Chord:* And it may either be greater or less than a *Semicircle*; as the Figure *SDG*, or *SMG*.

9. A SECTOR is a Figure included between Two *Radius*'s of the *Circle*, and that *Arch* of its *Periphery* where they *touch*, as the Figure *ACB:* And the Arch *AB* is the Measure of the *Angle* at *C*, included betwixt the *Radius's AC*. and *BC*.



Note, " All Angles of Sectors are call- " ed Angles at the Center of a Circle."

10. An ANGLE in the Segment of a *Circle* is that which is included between Two *Chords* that flow from one and the same Point in the *Periphery*, as at *D*, and meet with the Ends of another *Chord* Line, as at *F* and *G*.

That is, the *Angles* at *D*, at *F* and at *G*, are called *Angles* at the *Periphery*, or *Angles* standing on the Segment of a *Circle*,

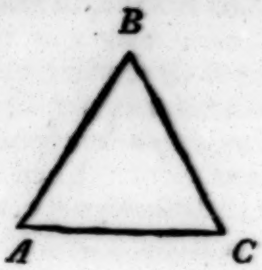
Sect. 3. *Of* TRIANGLES.

" There are two Kinds of Triangles, *viz*. Plain and Spherical; " but I shall not give any Definition of the Spherical, because " they more immediately relate to Astronomy."

1. A PLAIN TRIANGLE is a Figure whose *Area* is contained within the *Limits* of Three *Right Lines* called *Sides*, including Three *Angles*. And it may be divided, and takes its Name, ei- according to its *Sides* or *Angles*.

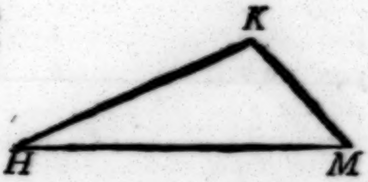
1. By

1. By its SIDES.



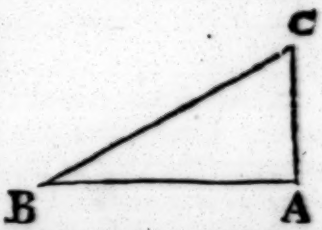
2. An EQUILATERAL TRIANGLE, is that which hath all its *Three Sides* equal; as the Figure *A B C.*
That is, $AB = BC = AC$.

3. An ISOSCELES TRIANGLE, is that which hath *only Two* of its *Sides equal,* as the Figure *BDG:* That is, $BD = DG$; but the *Third Side B G* may be either *greater* or *less,* as Occasion requires.



4. A SCALENE TRIANGLE, is that which hath all its *Three Sides* unequal;
such as the Figure *HKM.*

2. By its ANGLES.



5. A RIGHT-ANGLED *Triangle,* is that which hath one *Right Angle*; that is, when Two of its Sides are *Perpendicular* to each other, as *CA* is supposed to be to *BA.* Therefore the *Angle* at *A,* is a *Right Angle,* per Defin. 8. Sect. 1.

Note, " The longest Side of every Right-angled Triangle (as " *BC)* is called the Hypotenuse, and the longest of the other Two " Sides which include the Right Angle (as *B A)* is called the Base: " The third Side (as *CA)* is called the Cathetus or Perpendicu- " lar."

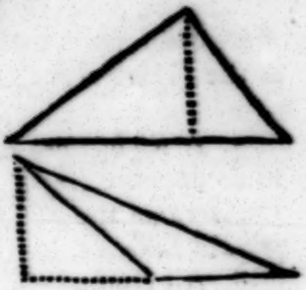
6. An OBTUSE-ANGLED *Triangle,* is that which hath one of its *Angles Obtuse,* and 'tis called an *Amblygonium Triangle.* Such is the third *Triangle HKM.*

7. An ACUTE-ANGLED *Triangle,* is that which hath all its *Angles Acute,* and 'tis called an *Oxygonium Triangle*; such are the first and second *Triangles ABC,* and *BDG.*

Note, " All Triangles that have not a Right Angle, whether they " are Acute, or Obtuse, are in general Terms, called Oblique Trian- " gles,

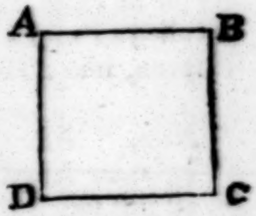
" gles, without any other Distinction, as before. And the long-
" est Side of every Oblique Triangle is usually called the *Base*;
" the other two are only called *Sides* or *Legs*."

8. The Altitude or Height of any *Plain Triangle*, is the Length of a *Right Line* let fall *perpendicular* from any of its *Angles*, upon the Side opposite to that *Angle* from whence it falls; and may be either within, or without the *Triangle*, as Occasion requires, being denoted by the *Two prick'd Lines*, in the *annexed Triangles*.



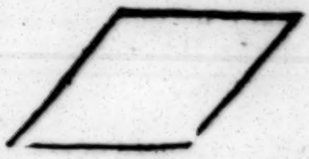
Sect. 4. *Of* Four-sided Figures, &c.

1. A Square is a plain *regular Figure*, whose *Area* is limited by Four equal *Sides* all *perpendicular* one to another.



That is, when $AB = BC = CD = DA$, and the *Angles* A, B, C, D are all *equal*, then it is usually called a *Geometrical Square*.

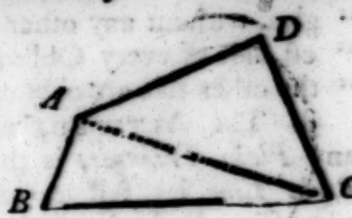
2. A Rhombus, or *Diamond-like Figure*, is that which hath *Four* equal *Sides*, but no *Right Angle*. That is, a *Rhombus* is a *Square* moved out of its *right Position*, as the *annexed Figure*.



3. A Rectangle, or a *Right-angled Parallelogram*; often called an *Oblong*, or *long Square*) is a *Figure* that hath *four Right-angles* and its *two opposite Sides* equal, *viz.* $BC = HD =$ and $BH = CD$.

4. A Rhomboides, is an *Oblique-angled Parallelogram*; that is, it is a *Parallelogram moved* out of its *right Position*, like the *annexed Figure*.

5. The Altitude or *Height* of any *Oblique angled Parallelogram*; *viz.* either of the *Rhombus* or *Rhomboides*, is a *Right-line* let fall *perpendicular* from any *Angle* upon the *Side* opposite to that *Angle*; and may either be within or without the *Figure*: As the *prick'd Lines* in the *annexed Figure*.



6. Every *Four-ſided Figure*, different from thoſe before-mentioned, is called a TRAPEZIUM.

That is, when it has neither *oppoſite Sides*, nor *oppoſite Angles equal*; as the *Figure* *A B C D*.

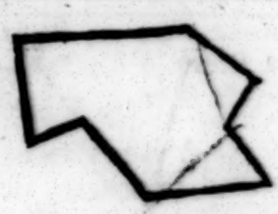
7. A *Right-line*, drawn from any *Angle* in a *Four-ſided Figure* to its oppoſite *Angle*, is called a DIAGONAL *Line*, and will *divide* the *Area* of the *Figure* into *two Triangles*, being *denoted* by the *prick'd Line AC* in the laſt *Figure*.

8. All *Right-lin'd Figures*, that have more than four Sides, are called *Polygons*, whether they be regular or irregular.

9. A REGULAR POLYGON is that which hath all its Sides equal, ſtanding at equal *Angles*, and is named according to the *Number* of its Sides (or *Angles*). That is, if it have five equal Sides, it is called a PENTAGON; if ſix equal Sides, it is call'd a HEXAGON; if ſeven, it is a HEPTAGON; if eight, 'tis an OCTAGON, &c.

Note, " All *regular Polygons* may be inſcribed in a Circle; that " is, their Angular Points, how many ſoever they have, will " all juſt touch the Circle's Periphery."

10. An IRREGULAR POLYGON is that *Figure* which hath many unequal Sides ſtanding at unequal *Angles* (like unto the annexed Figure, or otherwiſe); and of ſuch Kind of *Polygons* there are infinite Varieties, but they may all be reduced to regular Figures by drawing Diagonal Lines in them; as ſhall be ſhewed farther on.



Theſe are the moſt general and uſeful Definitions that concern plain or ſuperficial Geometry.

As for thoſe which relate to *Solids*, I thought it convenient to omit giving any Ancount of them in this Place, becauſe they would rather *puzzle* and *amuſe* the Learner, than improve him, until he has gained a competent Knowledge in the moſt uſeful *Theorems* concerning Superficies; for then thoſe Definitions may be more eaſily underſtood, and will help them to form a clearer Idea of their reſpective Solids, than it is poſſible to conceive of them before; and therefore I have reſerved thoſe Definitions until we come to the Fifth Part.

Sect. 5. *Of such* TERMS *as are generally used in Geometry.*

WHatsoever is proposed in *Geometry* will either be a PROBLEM or a THEOREM.

Both which *Euclid* includes in the general Term of Proposition.

A PROBLEM is that which proposes something to be done, and relates more immediately to practical than speculative *Geometry*; That is, 'tis generally of such a Nature, as to be performed by some known or commonly received Rules, without any Regard had to their Inventions or Demonstrations.

A THEOREM is when any commonly received Rule, or any new Proposition is required to be demonstrated, that so it may from thence forward become a certain Rule, to be relied upon in Practice when Occasion requires it. And therefore several Rules are often called *Theorems*, by which Operations in *Arithmetic*, and Conclusions in *Geometry*, are performed.

" *Note*, by DEMONSTRATION is understood the highest De-
" gree of *Proof* that human Reason is capable of attaining to, by
" a Train of Arguments deduced or drawn from such plain Axi-
" oms, and other Self-evident Truths, as cannot be denied by
" any one that considers them."

A COROLLARY, or CONSECTARY, is some *Consequent Truth* drawn or gained from any Demonstration.

A LEMMA is the Demonstration of some Premises laid down or proposed as preparative to obviate and shorten the Proof of the *Theorem* under Consideration.

A SCHOLIUM is a brief *Commentary* or *Observation* made upon some precedent Discourse.

N. B. " I advise the young Geometer to be very perfect in the
" Definitions, viz. Not to rest satisfied with a bare Remembrance
" of them; but, that he endeavour to gain a clear *Idea* or *Under-*
" *standing* of the Things defined; and for that Reason I have
" been fuller in every *Definition* than is usual.

" And, that he may know from whence most of the following
" *Problems* and *Theorems* contained in the two next Chapters are
" collected, I have all along cited the *Proposition* and Book of
" *Euclid's Elements* where they may be found.

" As for Instance; at *Problem* 1. there is (3. *e*. 1.) which shews
" that it is the Third Proposition in *Euclid*'s First Book. The
" like must be understood in the *Theorems*.

CHAP. II.

The First RUDIMENTS, or *Leading* and *Preparatory* PROBLEMS, in Plain GEOMETRY. .

" IN order to perform the following *Problems*, the young Geo-
" meter ought to be provided wsth a thin streight Ruler, made
" either of *Brass* or *Box-wood*, and two Pair of very good Com-
" passes, viz. one Pair called Three-pointed Compasses, be-
" ing very useful for drawing of Figures or Schemes, either
" with Black Lead or Ink; and one Pair of plain Compasses
" with very fine Points, to measure and set off Distances; also he
" should have a very good Steel *Drawing-Pen:* And then he may
" proceed to the Work with this Caution; that he ought to make
" himself Master of one *Problem* before he undertakes the next:
" That is, he ought to understand the Design, and, as far as he
" can, the Reason of every *Problem*, as well as how to do it; and
" then a little Practice will render them very easy, they being all
" grounded upon these following *Postulates*."

POSTULATES OF PETITIONS.

1. That a *Right-line* may be drawn from any one given *Point* to another.

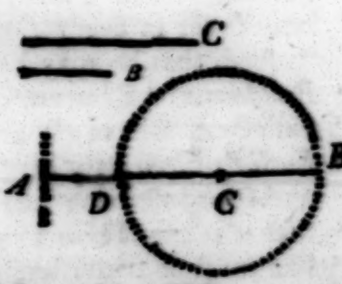
2. That a *Right-line* may be produced, encreased, or made longer from either of its Ends.

3. That upon any given *Point* (or *Center*) and with any given *Distance* (*viz.* with any RADIUS) a *Circle* may be described.

PROBLEM I.

Two Right-lines being given, to find their Sum and Difference (3. *e.* 1.)

Let the given *Lines* be $\left\{ \begin{array}{l} A \\ C \end{array} \right.$



Make the shortest Line, as *CB*, *Radius*, and with it describe a *Circle:* From its Center *C* set off the other *Line AC*, and join *ACB* with a *Right-line*. Then will $AB = AC + CB$; and $AD = AC - CB$; as was required.

PRO-

PROBLEM II.

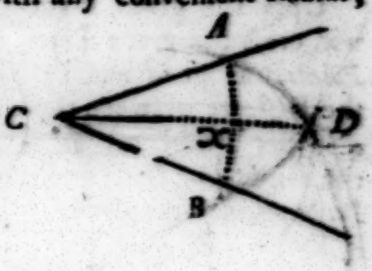
To bisect, or divide a Right-line given (*as AB*) *into two equal Parts* (10. *e.* 1.)

From both Ends of the given *Line* (*viz. A* and *B*) with any *Radius* greater than half its *Length*, describe Two *Arches* that may *cross* each other in two *Points*, as at *D* and *F*; then join those *Points DF* with a *Right-line*, and it will *bisect* the *Line AB* in the Middle, at *C*; *viz.* it will make $AC = CB$; as was required.

PROBLEM III.

To bisect a Right-lin'd Angle given, into two equal Angles. (9. *e.* 1.)

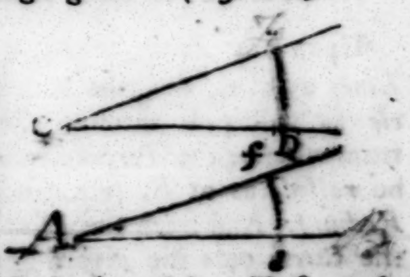
Upon the *Angular Point*, as at *C*, with any convenient *Radius*, describe an *Arch* as *AB*; and from those *Points A* and *B*, describe two equal *Arches crossing* each other, as at *D*; then join the *Points C* and *D* with a *Right-line*, and it will *bisect* the *Arch AB*, and consequently the *Angle*; as was required.



PROBLEM IV.

At a Point A, in a Right-line given AB, to make a Right-lin'd Angle equal to a Right-lin'd Angle given C. (23. *e.* 1.)

Upon the given *Angular Point C* describe an *Arch*, as *FD*, (making *CD* any *Radius* at Pleasure) and with the same *Radius* describe the like *Arch* upon the given *Point A*, as *fd*; that is, make the *Arch fd* equal to the *Arch FD*; Then join the *Points A* and *f* with a *Right-line*, and it will form the Angle required.

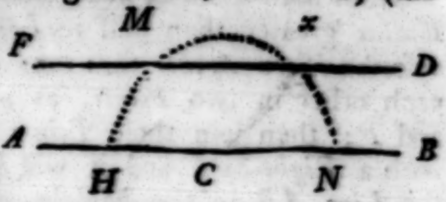


PRO-

PROBLEM V.

To draw a Right-line, as FD, parallel to a given Right-line AB, that ſhall paſs through any aſſigned Point. as at x, viz. *at any Diſtance required.* (31. *e.* 1.)

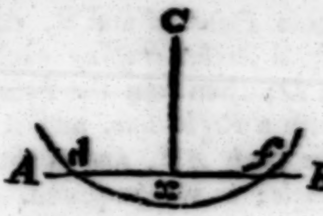
Take any convenient Point in the given *Line,* as at *C,* (the farther off *x* the better;) make *C x Radius,* and with it upon the *Point C,* deſcribe a *Semicircle,* as *H M x N*; then make the *Arch H M* equal to the *Arch x N*; thro' the *Points M* and *x* draw the *Right-line F D,* and it will be *parallel* to the *Line AC,* as was required.



PROBLEM VI.

To let fall a Perpendicular, *as C x, upon a given Right-line A B, from any aſſigned Point that is not in it, as from* C. (12. *e.* 1.)

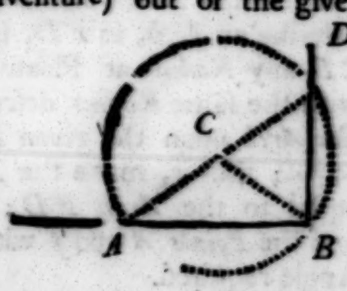
Upon the given Point *C* deſcribe ſuch an *Arch* of a *Circle* as will croſs the given Line *A B* in two *Points*; as at *d* and *f*; Then *biſect* the *Diſtance* between thoſe two *Points d f* (per Probl. 2.) as at *x*. Draw the *Right-line Cx,* and it will be the *Perpendicular* required.



PROBLEM VII.

To erect or raiſe a Perpendicular *upon the End of any given Right-Line, as at A; or upon any other Point aſſigned in it.* (11. *e.* 1.)

Upon any *Point* (taken at an Adventure) out of the given *Line,* as at *C,* deſcribe ſuch a *Circle* as will paſs through the *Point* from whence the *Perpendicular* muſt be *raiſed,* as at *B,* (viz. make *C B Radius*): And from the *Point* where the *Circle* cuts the given *Line,* as at *A,* draw the *Circle's Diameter A C D*; then from the *Point D* draw the *Right-line D B,* and it will be the *Perpendicular* as was required.

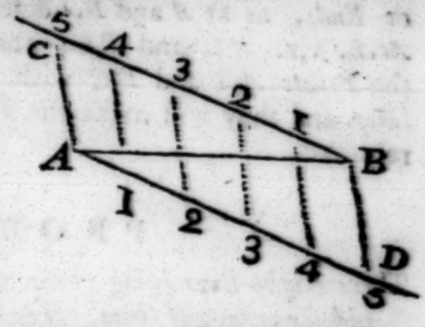


PRO-

PROBLEM VIII.

To divide any given Right-line, as AB, into any proposed Number of equal Parts. (10. e. 6.)

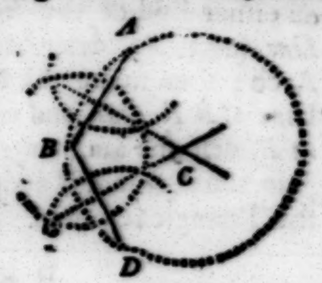
At the extream Points (or Ends) of the given Line, as at *A* and *B*, make two equal *Angles* (by Prob. 4.) continuing their Sides *AD* and *BC* to any sufficient *Length*; then upon those Sides, beginning at the Points *A* and *B*, set off the proposed *Number* of equal Parts (*suppose 'em* 5.) If *Right-lines* be drawn (*cross the given Line*) from one Point to the other, as in the annexed Figure, those *Lines* will divide the given *Line AB* into the *Number* of equal Parts required.



PROBLEM IX.

To describe a Circle that shall pass (or cut) through any three Points given, not lying in a Right-line, as at the Points A B D.

Join the Points *AB* and *BD* with *Right-lines*; then bisect both those *Lines* (per Problem 2.) the Point where the bisecting Lines meet, as at *C*, will be the *Center* of the *Circle* required.



The Work of this Problem being well understood, it will be easy to perform the two following, without any *Scheme*, viz.

1. *To find the Center of any Circle given.* (1. e. 3.)

By the last Problem it is plain, that if three Points be any where taken in the given *Circle's Periphery*, as at *A*, *B*, *D*, the *Center* of that Circle may be found as before.

2. *If a Segment of any Circle be given, to complete or describe the whole Circle.*

This may be done by taking any *three Points* in the given *Segment's* Arch, and then proceed as before.

PRO-

PROBLEM X.

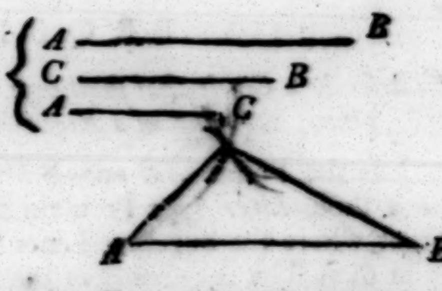
Upon a Right-line given, as AB, to describe an Equilateral Triangle. (1. e. 1.)

Make the given Line *Radius*, and with it, upon each of its extream *Points* or *Ends*, as at *A* and *B*, describe an *Arch*, viz. *AC* and *BC*; then join the *Points AC* and *BC* with *Right-lines*, and they will make the *Triangle* required.

PROBLEM XI.

Three Right-lines being given, to form them into a Triangle, (provided any two of them, taken together, be longer than the Third) (22. e. 1.)

Let the given *Lines* be

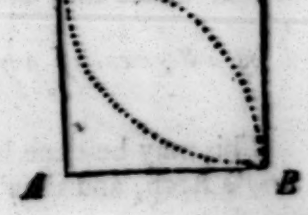


Make either of the *shorter Lines* (as *AC*) *Radius*, and upon either End of the *longest Line* (as at *A*) describe an Arch; then make the other Line *CB Radius*, and upon the other End of the longest *Side* (as at *B*) describe another Arch, to cross the first Arch (as at *C*): Join the Points *CA* and *CB* with Right-lines, and they will form the Triangle required.

PROBLEM XII.

Upon a given Right-line, as AB, to form a Square. (46. e. 1.)

Upon one End of the given Line, as at *B*, erect the *Perpendicular BD*, equal in Length with the given Line, viz. make $BD = AB$; that being done, make the given Line *Radius*, and upon the Points *A* and *D* describe equal Arches to cross each other, as at *C*; then join the Points *CA* and *CD* with Right-lines, and they will form the *Square* required.



PRO-

PROBLEM XIII.

Two unequal Right-lines being given, to form or make of them a Right-angled Parallelogram.

Let the given *Lines* be { *A*————————*B*
B————*C*

Upon one End of the *longest Line*, as at *B*, erect a *Perpendicular* of the same Length with the *shortest Line B C*; then from the *Point C* draw a Line Parallel, and of the same Length, to *A B*; *viz.* make $DC = AB$: Join *D A* with a *Right-line*, and it will form the Oblong or Parallelogram required.

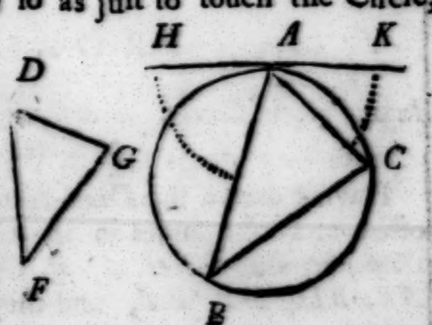
As for *Rhombus*'s and *Rhomboides*'s, to wit, Oblique-angled Parallelograms, they are made, or described, after the same Manner with the two last Figures; only instead of erecting the Perpendiculars, you must set off their given Angles, and then proceed to draw their Sides parallel, &c. as before.

PROBLEM XIV.

In any given Circle, to inscribe or make a Triangle, whose Angles shall be equal to the Angles of a given Triangle; as the Triangle F D G, (2. *e.* 4.)

Note, " Any Right-lined Figure is said to be inscribed in a " Circle, when all the Angular Points of that Figure do just touch " the Circle's Periphery.

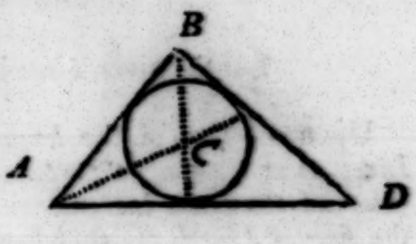
Draw any Right-line (as *H K*) so as just to touch the Circle, as at *A*; then make the Angle *K A C* equal to any one Angle of the given Triangle, as *D F G*; and the Angle *H A B* equal to another Angle of the Triangle, as *D G F*; then will the Angle *B A C* be equal to the Angle *F D G*. Join the *Points B* and *C* with a Right-line, and 'twill form the Triangle required.



PROBLEM XV.

In any given Triangle, as ABD, to describe a Circle that shall touch all its Sides. (4. e. 4.)

Bisect any two Angles of the Triangle, as *A* and *B*, and where the bisecting Lines meet (as at *C)* will be the Center of the Circle required; and its *Radius* will be the nearest Distance to the Sides of the Triangle.



PROBLEM XVI.

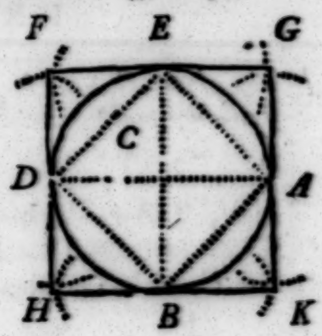
To describe a Circle about any given Triangle. (5. e. 4.)

This *Problem* is performed in all Respects like the Ninth, *viz.* by bisecting any two *Sides* of the given Triangle; the Point, where those bisecting Lines meet, will be the Center of the Circle required.

PROBLEM XVII.

X *To describe a Square about any given Circle.* (7. e. 4.)

Draw two *Diameters* in the given Circle (as *DA* and *EB)* crossing at Right Angles in the Center *C*; and, with the Circle's *Radius CA*, describe from the extream *Points* of those Diameters, viz. *A, B, D, E,* cross Arches, as at *F, G, H, K*; then join those *Points* where the Arches cross with *Right-lines,* and they will form the *Square* required.



PROBLEM XVIII.

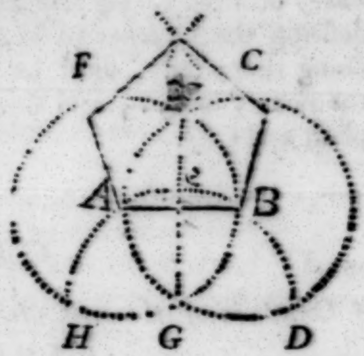
In any given Circle, to describe the largest Square it can contain. (6. e. 4.)

Having drawn the *Diameters,* as *DA* and *EB,* bisecting each other at Right-angles in the Center *C,* (as in the last Scheme); then join the *Points A, B, C,* and *E,* with Right-lines, viz. *AB, BD, DE, EA,* and they will be Sides of the *Square* required.

PRO-

PROBLEM XIX.

Upon any given Right-line, as AB, to describe a regular Pentagon, *or Five-sided* Polygon.



Make the given Line *Radius*, and upon each End of it describe a Circle; and through those Points where the Circles cross each other (as at *G x*) draw the Right-line *G e x*: Upon the Point *G* with the same *Radius* describe the Arch *H A e B D*, and laying a Ruler upon the Points *D*, *e*, mark where it *crosses* the other Circle, as at *F*. Again, lay the Ruler upon the Points *H*, *e*, and mark where it *crosses* the other Circle, as at *C*: Then from the Points *F* and *C* (with the same *Radius* as before) describe cross Arches, as at *K*: Join the Points *AF*, *FK*, *KC*, and *CB*, with Right-lines, and they will form the *Pentagon* required, *viz.* $AF = FK = KC = CB = AB$; and the Angles at *A*, *B*, *C*, *K*, *F* will be equal.

PROBLEM XX.

In any given Circle, to describe a regular Pentagon.
(11. *e*. 4. & 10. *e*. 3.)

Or, in general Terms, to describe any regular Polygon *in a Circle*.

Draw the Circle's Diameter *D A*, and divide it into as many equal Parts as the proposed *Polygon* hath Sides; then make the whole Diameter a *Radius*, and describe the two Arches *C A* and *C D*. If a Right-line be drawn from the Point *C*, through the second of those equal Parts in the Diameter, as at 2, it will assign a Point in the opposite Semicircle's *Periphery*, as at *B*. Join *D B* with a Right-line, and it will be the Side of the *Pentagon* required.

These twenty *Problems* are sufficient to exercise the young Practitioner, and bring his Hand to the right Management of a *Ruler* and *Compasses*, wherein I would advise him to be very ready and exact.

As to the Reason why such Lines must be so drawn as directed at each *Problem*, that, I presume, will fully and clearly appear from the following *Theorems*; and therefore I have *(for Brevity's Sake)* omitted giving any *Demonstration* of them in this Chapter, desiring the Learner to be satisfied with the bare Knowledge of doing them only, until he hath fully considered the Contents of the next Chapter; and then I doubt not but all will appear very plain and easy.

CHAP. III.

A Collection of the most useful THEOREMS *in plain Geometry* DEMONSTRATED.

Note, *In order to shorten several of the following Demonstrations, it will be necessary to premise, that*

1. THE Periphery (or Circumference) of every Circle (whether great or small) is supposed to be divided into 360 equal parts, called *Degrees*; and every one of those *Degrees* are divided into 60 *equal Parts*, called *Minutes*, &c.

2. All *Angles* are measured by the Arch of a Circle described upon the *angular Point* (*See Defin.* 9. *Page* 287.) and are esteemed *greater* or *less*, according to the Number of Degrees contained in that *Arch*.

3. A *Quadrant*, or *Quarter-part* of any Circle, is always 20 Degrees, being the *Measure* of a *Right-angle (Defin. 6. p.* 287.) and a *Semicircle* is 180 Degrees, being the *Measure* of *two Right-angles*.

4. *Equal Arches* of a *Circle*, or of equal Circles, measure equal *Angles*.

To those five general *Axioms* already laid down in *Page* 146, (which I here suppose the Reader to be very well acquainted with) it will be convenient to understand these following, which begin their Number where the other ended.

AXIOMS.

Axioms.

5. Every whole Thing is Greater than its Part.
That is, the whole Line *AB* is *greater* than its Part *A c*, &c. } *A* ——— | *c* ——— *B*
The same is to be understood of *Superficies*'s and *Solids*.

6. Every whole is Equal to all its Parts taken together.
That is, the whole Line *A B* is equal to its *Parts* $AC + cd + de + eB$. } *A* — | *c* — | *d* — | *e* — *B*
The same is also true in *Superficies*'s and *Solids*.

7. Those Things which, being laid one upon another, do agree or meet in all their Parts, are equal one to the other.

But the Converse of this *Axiom*, to wit, that *equal Things* being laid one upon the other will meet, is only true in *Lines* and *Angles*, but not in *Superficies*'s, unless they be alike, *viz.* of the same *Figure* or Form: As for Instance, a Circle may be equal in Area to a Square; but if they are laid one upon the other, it is plain they cannot meet in all their *Parts*, because they are *unlike Figures*. Also, a *Parallelogram* and a *Triangle* may be equal in their *Area*'s one to another, and both of them may be equal to a *Square*; but if they are *laid* one upon the other, they will not *meet* in all their *Parts*, &c.

Note, *Besides the* Characters *already explained in* Part I, *and in other Places of this Tract, these following are added.*

Viz. $<$ denotes an Angle in general, and $<<$ signifies Angles; △ signifies a Triangle; □ signifies a Square, and ▭ denotes a Parallelogram. And when an Angle is denoted by any *three Letters* (as, *A B C*) the middle *Letter* (as *B*) always denotes the Angular Point; and the other *two Letters* (*as A B* and *B C*) denote the *Lines* or *Sides* of the Triangle which includes that Angle.

These Things being premised, the young *Geometer* may proceed to the *Demonstrations* of the following *Theorems*; wherein he may perceive an absolute Necessity of being well versed in several Things that have been already delivered: And also it will be very *advantageous* to store up several useful *Corollaries* and *Lemma*'s, as they become *discovered Truths*: For it often happens, that a *Proposition* cannot be clearly demonstrated *a priori*, or of itself, without a great Deal of Trouble; therefore it will be useful to have *Recourse* to those *Truths* that may be assisting in the *Demonstration* then in Hand.

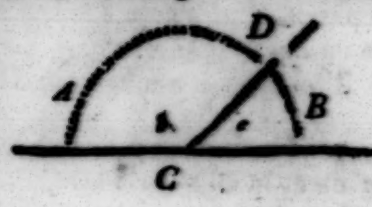
THEO-

THEOREM I.

If a Right-line stand upon (or meet with) another Right-line, and make Angles with it, they will either be two Right-angles, or two Angles equal to two Right-angles. (13. *e.* 1.)

Demonstration.

Suppose the Lines to be *AB* and *DC*, meeting in the *Point* at *C*: Upon *C* describe any Circle at pleasure; then will the Arch *AD* be the *Measure* of the $< b$, and the Arch *DB* the *Measure* of $< e$; but the Arches $AD + DB = 180°$, *viz.* they compleat the *Semicircle*. Consequently the $< b + < e = 180°$. *Which was to be proved.*



Corollaries.

1. Hence it follows, that if the $< b = 90°$ the $< e = 90°$; but if $< b$ be *obtuse*, then the $< e$ will be *acute*, &c.

From hence it will be easy to conceive, that if several Right-lines stand upon, or meet with any Right-line at one and the same Point, and on the same Side, then all the Angles taken together will be $= 180°$, *viz.* Two Right-angles.

THEOREM II.

If two Right Lines intersect (i. e. *cut or cross*) *each other, the two opposite Angles will be equal.* (15. *e.* 1.)

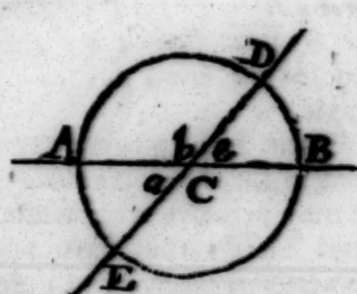
Demonstration.

Let the *two Lines* be *A B* and *D E*, intersecting each other in the Center *C*.

Then $< b + < e = 180°$
And $< b + < a = 180°$ } *per* last.

Consequently $< b + < e = < b + < a$, *per Axiom* 5.

Substract $< b$ on both Sides of the *Equation*, and it will leave $< e = < a$.



Again, $< b + < e = 180°$, as before; and $< e + < C = 180°$, consequently $< e + C = < b + < e$. *Substract* $< e$, and then $< C = < b$. Q. E. D.

Corol.

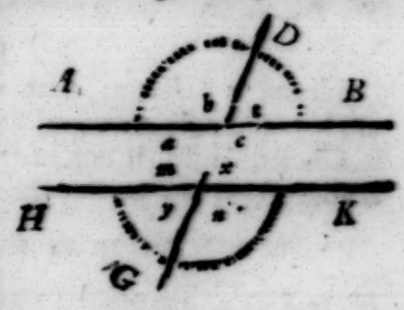
Corollary.

From hence it is evident, that if two Lines intersect each other, they will make four Angles; which, being taken together, will always be equal to four Right-angles.

THEOREM III.

If a Right-line cut (or cross) two parallel Lines, it will make the opposite Angles equal one to another. (29. *e.* 1.)

Suppose the two Lines *AB* and *HK* to be parallel, and the Right-line *D G* to cut them both at *C* and *n:* Upon the Point *C* (with *any Radius)* describe a Semicircle; and with the same *Radius*, upon the Point at *n*, describe another Semicircle opposite to the first, as in the Figure. Then it is plain, and I suppose very easy to conceive, that if the Center *C* were moved along upon the Line *DG*, until it came to the Center at *n*, the two Lines *AB* and *HK* would *meet* and concur, *viz.* become one Line *(for parallel Lines are as it were but one broad Line)*. Consequently the two Semicircles would also meet, and become one entire Circle, like to that in the last *Demonstration*.



And therefore the $<y=<x=<a=<e$ } { as before, *per last*
And $<m=<n=<b=<c$ } { *Theorem.*

Q. E. D.

Corollary.

Hence it follows, that if three, four, or ever so many Parallel Lines, are cut or crossed by one Right-line, all their opposite Angles will be equal.

THEOREM IV.

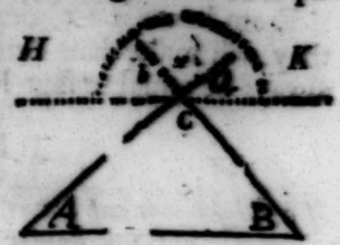
The three Angles of every plain Triangle are equal to two Right-angles. (32. *e.* 1.)

Consequently, any two Angles of any plain Triangle must needs be less than two Right-angles. (17. *e.* 1.)

DEMON-

DEMONSTRATION.

Let the $\triangle$ *ABC* be proposed; draw the Right-line *HK* parallel to the Side *AB*, just touching the Vertical Angle *C*; and upon the same Angular Point *C* describe any Semicircle, and produce the Sides *AC* and *BC* to its *Periphery*. Then will $<b=<B$, $<a=<A$, and $<x=<C$, *per* last *Theorem*. But $<b+<a+<x=180°$, or two Right-angles: Consequently $<B+<A+<C=180°$ per *Axiom* 5. Q. E. D.



Corollary.

Hence it follows, that the two *acute* Angles of every Right-angled Triangle are equal to a Right-angle, of 90°.

Consequently, if one of the *acute* Angles be given, the other is also given, *viz.* 90°— the given < leaves the other <.

THEOREM V.

If one Side of any plain Triangle be continued or produced beyond, or out of the Triangle, the outward Angle will always be equal to the two inward opposite Angles. (32. *e.* 1.)

DEMONSTRATION.

Let the Side *AB* of the $\triangle$ *ABC* be produced out of the $\triangle$, suppose to *D*, &c. as in the Figure. Then $<z=<A+<C$, for the $<B+<z=180°$ per *Theorem* 1. and the $<B+<A+<C=180$, *per* last *Theorem*. Therefore $<B+<z=<B+<A+<C$, per Axiom 5. *Substract* $<B$ on both Sides the *Equation*, and it will leave $<z=<A+<C$ (*per Axiom* 2.) Q. E. D.

Consequently, the outward Angle (at z) of any plain Triangle, must needs be greater than either of the inward opposite Angles, viz. *greater than* $<A$, *or* $<C$ (16. *e.* 1.)

Corollary.

Hence it follows, that if one Angle of any plain Triangle be given, the *Sum* of the other two Angles is also given; for 180°— the given < = the other two < <.

THEO-

THEOREM VI.

In every plain Triangle, equal Sides subtend (i. e. *are opposite to*) *equal Angles.* (5. *e.* 1.)
Consequently, equal Angles are subtended by equal Sides (6. *e.* 1.)

Demonstration.

Suppose the $\triangle BCD$ to be an *Isosceles* $\triangle$; that is, let $BC = CD$. Bisect the $< C$, or (*which is all one*) make CA *perpendicular* to BD; then will the $<<$ on each Side of it (*viz.* $< BAC$ and $< DAC$) be Right-angles.

Therefore $\left\}\begin{array}{l} \frac{1}{2} < C + < B = 90^\circ \\ \frac{1}{2} < C + < D = 90^\circ \end{array}\right\}$ per *Corol.* to *Theorem* 4.
Consequently, $\frac{1}{2} < C + < B = \frac{1}{2} < C + < D$, per *Axiom* 5. *Substract* $\frac{1}{2} < C$ from both Sides of the *Equation*, and it will leave $< B = < D$, per *Axiom* 2. Q. E. D.

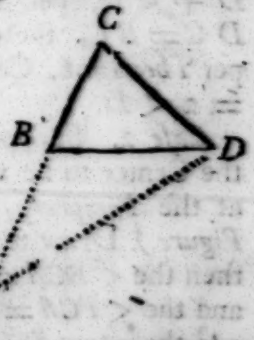
Corollary.

From hence it follows, that the *three Angles* of an *Equilateral Triangle* are equal one to another.

THEOREM VII.

In every plain Triangle, the longest Side subtends the greatest Angle. (18. *e.* 1.)
Consequently, the greatest Angle of any plain Triangle is subtended by the longest Side.

This *Theorem* is evident by inspection only: For, let one of the Sides of any *plain Triangle* (as CB) be produced, suppose to E; join DE with a *Right-line*; then it is evident, that because CE is now made longer than the Side BC, therefore the $<$ at D is become *larger* than it was before by the $< BDE$: And it is plain, the longer the Side CE had been made, the $<$ at D would have been the more enlarged.



THEOREM VIII.

If the Sides of two Triangles are equal, the Angles opposite to those equal Sides will be equal. (8. *e.* 1.)

The Truth of this *Theorem* is evident by the two included *Triangles* in the 6th *Theorem*, for they have their respective Sides equal, *viz.* $BC = CD$, $BA = DA$, and CA common to both *Triangles*. And it is there proved, that the < opposite to those equal Sides are equal, &c. which needs no further *Proof*.

Note, *The Converse of this* Theorem *holds not true; for the Angles of two Triangles may be equal, and their opposite or subtending Sides unequal; as will appear at* Theorem XII.

Corollary.

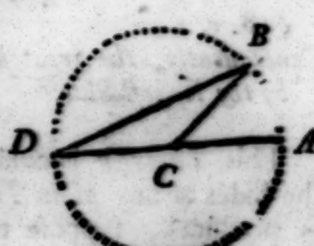
Hence it follows, that *Triangles mutually equilateral* are also *mutually equiangular*; and,

That *Triangles mutually equilateral* are equal one to another. (4. & 26. *e.* 1.)

THEOREM IX.

An Angle at the Center of any Circle is always double to the Angle at the Periphery, when both the Angles stand upon the same Arch. (20. *e.* 3.) This *Theorem* hath three Varieties or Cases.

DEMONSTRATION.



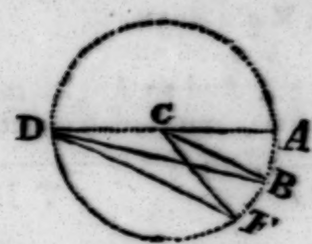
Case 1. Let the Diameter DA, and the Line DB, be the two Lines which form the $<D$ at the *Periphery*; draw the *Radius* BC, then $<BCA = <D + <B$, per *Th.* 5, and because $DC = BC$, therefore $<D = <B$, per *Theorem* 6. Consequently $<BCA = 2<D$.

Case 2. Suppose the $<BCF$ at the Center to be within the $<BDF$ at the *Periphery*, (*as in the annexed Figure.*) Draw the Diameter DA; then the $\left.\begin{array}{l}<BCA=2<BDA\\ \text{and the } <FCA=2<FDA\end{array}\right\}$ per *Case* 1. add these two *Equations* together.

Then

Then will $<BCA + <FCA = 2<BDA + 2<FDA$, per *Ax.* 1. But $<BCA + <FCA = <BCF$, and $2<BDA + 2<FDA = 2<BDF$. Consequently $<BCF = 2<BDF$.

Case 3. Again, suppose the $<BCF$ at the Center to be out of the $<BDF$ at the Periphery. From the Angular Point D at the Periphery draw the Diameter DA.



Then $\left.\begin{array}{l}<FCA = 2<FDA \\ \text{and } <BCA = 2<BDA\end{array}\right\}$ per Case 1.

Substract this last *Equation* from the other, and it will leave $<FCA - <BCA = 2<FDA - 2<BDA$, per *Axiom* 2. But $<FCA - <BCA = <FCB$, and $2<FDA - 2<BDA = 2<FDB$: Consequently $<FCB = 2<FDB$. Q. E. D.

Corollary.

Hence 'tis evident, that all Angles at the Periphery, which stand on the same *Segment* or *Arch* of a Circle, or upon equal Arches, are equal one to another. (21. *e.* 3.)

THEOREM X.

An Angle in a Semicircle is a Right-angle. (31. *e.* 3.) That is, *if the Diameter of any* Circle *be the Side of a Triangle, and the Angle opposite to that Side be any where in the* Circle's *Periphery, it will be a Right-angle.*

Demonstration.

Let DA be the Diameter, and DBA the Triangle, the $<B = 90°$. Draw the Radius BC, then is the $<DBA = <D + <A$. For $<CBD = <D$, and $<CBA = <A$, per *Theorem* 6. Therefore $<DBA = <CBD + <CBA$, per *Axiom* 5. Again $<DBA + <D + <A = 180°$. per *Theorem* 4. Consequently, $<DBA = 90°$ or a Right-angle. Q. E. D.

Corollaries.

1. Hence it will be easy to conceive, that an Angle made in any Segment less than a Semicircle will be *obtuse*, or greater than a Right-angle.

2. And an Angle, made in any Segment greater than a Semicircle, must consequently be *acute*.

THEOREM XI.

In any Right-angled Triangle, the Square which is made of the Hypothenuse, or Side subtending the Right-angle, is equal to both the Squares which are made of the Sides including the Right-angle, (47. *e.* 1.)

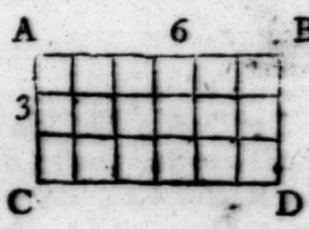
There are several Ways of demonstrating this noble and useful *Theorem*, but, I presume, none more easy to be understood by a *Learner* than that which I shall here propose: And, in order thereto, 'twill be necessary to premise the following *Lemma*'s.

Lemma 1.

A *Right-line* is said to be *multiplied* with a *Right-line*, when either a *Square*, or other *Right-angled Parallelogram* is made of the *two Lines*.

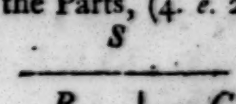
That is, the *Area* of any *Right-angled Parallelogram* is equal to the *Product* of those *Numbers* which express the *Measure* of its Sides.

Thus, if $AB = 6$ Inches, and $AC = 3$ Inches. Then $AB \times AC = 6 \times 3 = 18$ square Inches; which is the *Area* of the Parallelogram $ABCD$,



Lemma 2.

If a *Right-line* be any Way cut into *two Parts*, the *Square* of the whole *Line* will be equal to the *Squares* of each Part, and a double *Rectangle* or *Parallelogram* made of both the Parts, (4. *e.* 2.) that is, if the *Line* S be cut into the two Parts B and C; then is $S = B + C$: But if both the Sides of the *Equation* be involved, it will be $SS = BB + 2\,BC + CC$.



Lemma

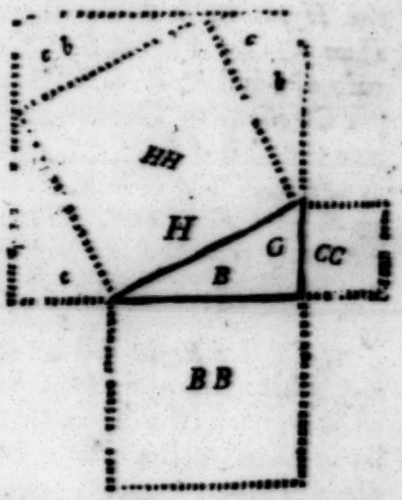
Lemma 3.

The *Area* of every *Right-angled Triangle* is half the *Parallelogram* made of its Base and Perpendicular.

For $B \times C =$ the *Area* of the whole *Parallelogram*, by the first *Lemma*. And $\triangle$ $BCH + \triangle\, bcH =$ the *Parallelogram*; but $B = b$, and $C = c$. Therefore $\frac{1}{2}\ B \times C$ $=$ the *Area* of each $\triangle$, *viz.* $\frac{1}{2} B \times C + \frac{1}{2}\, b \times c = B \times C$.

These Things being premised, let us suppose the Triangle BCH to be a *Right-angled Triangle*, *viz.* the Side C perpendicular to the Side B; then will $BB + CC = HH$.

Demonstration.



Make a *Square* whose Side is $= B + C$, and draw the included *Square* whose Side is $= H$, as in the Scheme: Then will the *Area* of the *great Square* be equal to the *Area* of the four *Triangles* $+ HH$; but the *Area* of each $\triangle = \frac{1}{2}\, BC$, per *Lemma* 3. Therefore the 4 $\triangle$'s $= \frac{1}{2}\, BC \times 4 = 2\, BC$, consequently, the *Area* of the great *Square* is $HH + 2BC$. *Involve* $B + C$, and it will be $BB + 2\, BC + CC =$ the *Area* of the great *Square*; per *Lemma* 3. Consequently, $HH + 2\, BC = BB + 2\, BC + CC$, per *Axiom* 5. Substract $2\, BC$ from both Sides of the *Equation*, and there will *remain* $HH = BB + CC$.

To illustrate this *Theorem* by *Numbers*, let us
Suppose $C = 3$. $B = 4$. and $H = 5$.
Then will $CC = 9$. $BB = 16$. and $HH = 25$.
Consequently, $BB + CC = HH = 16 + 9 = 25$.

Consectary.

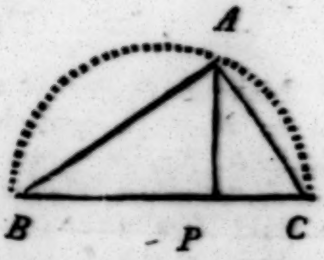
From this admirable *Theorem* (*said to be the first invented by* Pythagoras) is deduced the *Method* of adding and substracting Squares, Parallelograms, Circles, *&c.*

THEO-

THEOREM XII.

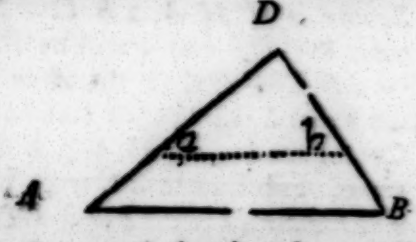
In any Right-angled Triangle, a Perpendicular being let fall from the Right-angle upon the Hypothenuse will divide the Triangle into two Right-angled Triangles, which will be both similar (or alike) to the first Triangle, and to each other. (8. *e.* 6.)

Note, *All plain Triangles are said to be similar* (i. e. *alike) when each single Angle in one of the Triangles is equal to each single Angle of the other; but if any two single Angles of one Triangle are equal to two single Angles of the other, the third Angle will be equal. Per* Theo. 4.



1. In the Right-angled $\triangle$ *BAC*, let *AP* be supposed *perpendicular* to the *Hypothenuse* *B C*; then $< BAP = < C$. For $< BAP + < B = 90°$, and $< B + < C = 90°$, per *Corollary* to *Theorem* 4. Therefore the $< BAP = < C$, per *Axiom* 5. again, $< PAC + < C = 90$, and $< B + < C = 90°$. Therefore $< PAC = < B$, &c. Consequently the $\triangle$ *BAP* is alike to the $\triangle$ *ACP*; and each is alike to the whole $\triangle$ *BAC*.

2. Or if a *Right-line* be drawn parallel to one of the Sides of any *plain Triangle*, (*viz.* within it) it will cut off a Triangle similar or alike to the whole Triangle. Thus:



In the $\triangle$ *ABD* draw the *Right-line a b* parallel to the Side *AB*; then will the included $\triangle$ *a D b* be like the $\triangle$ *ADB*: For $< a = < A$ and $< b = < B$, per *Theorem* 3; and $< D$ is common to both the Triangles; *Ergo*, &c.

THEOREM XIII.

If two Triangles are alike, their like Sides will be proportional.

That is, those Sides which subtend the equal Angles, as also those Sides which are about the equal Angles, will be *proportional* to each other; and consequently, if any *two Triangles* have their Sides proportional, their Angles are equal. (4, 5, 6, 7. *e.* 6.)

DEMON-

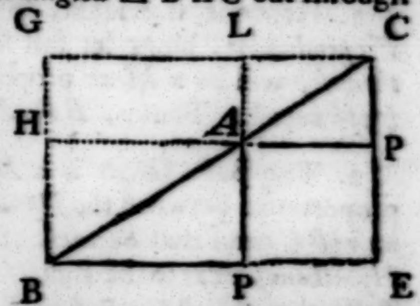
Demonstration.

Let the *similar Triangles* in the *Scheme* of the last *Theorem* be here proposed again.

Then it will be $BP : AP :: AP : CP$, according to this *Theorem*. *Ergo* $BP \times CP = AP \times AP$.

First.

Let us suppose the aforesaid *Right-angled* $\triangle\ BAC$ cut through the Perpendicular AP, and there opened until the Sides BA and CA become one *Right-line*. Let the Sides BP and CP be continued until they meet in E; then compleat the Parallelograms by drawing the parallel Lines GLC, HAP, GHB, and LAP, as in the *Figure*.

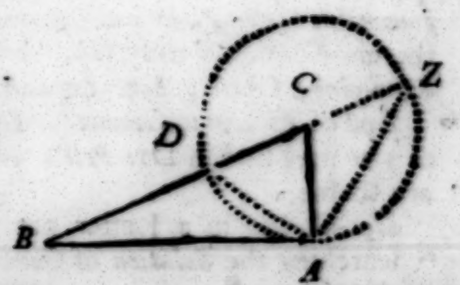


Then it is evident, that the $\triangle BHA = \triangle BPA$, and the $\triangle CPA = \triangle CLA$; also that the $\triangle BEC = \triangle BGC$, because all their respective Sides are *equal*.

But the $\triangle BHA + \triangle CLA + \square HGLA = \triangle BPA, + \triangle CPA + \square APEP$. Now, if from both *Sides* of this *Equation* there be substracted the equal *Triangles*, there will remain $\square\ HGLA = \square\ APEP$. But $\square\ HGLA = BP \times CP$, and $\square\ APEP = AP \times AP$. Consequently $BP : AP :: AP : CP$. Which was to be proved.

Or otherwise, thus:

Suppose the $\triangle\ BAC$ to be *Right-angled* at A: Upon the $<$ Point C, with the *Radius* CA describe a Circle, and continue the *Hypothenuse* BC to Z; join ZA and AD with *Right-lines*; then will the $\triangle\ BAD$ be like to the $\triangle\ BZA$. For $\triangle\ DAB + < DAC = 90°$, by *Construction*. And $< ZAC + < DAC = 90°$, by *Theorem* X. Therefore $< DAB + < DAC = < ZAC + < DAC$. By *Axiom* 5. *substract* $< DAC$ from both Sides of the *Equation*, and there will remain $< DAB = < ZAC$. But $< ZAC = < CZA$, by *Theorem* 6.



And

And $< B$ is common to both Triangles. Therefore $< BDA = < BAZ$, by *Theorem* 4, consequently $\triangle BAD$ is like to $\triangle BZA$.

Let the *Sides* $\left\{\begin{array}{l} BA = b \\ BC = h \\ CA = c \end{array}\right\}$ $\left\{\begin{array}{l}\text{Then } bb + cc = hh, \text{ by } \textit{Theorem} \\ \text{11. Consequently } bb = hh - cc, \\ \text{which gives the following Analogy,}\end{array}\right.$

Viz. $h : b + c :: b - c : h$; that is, $BA : BZ :: BD : BA$.

Q. E. D.

Corollaries.

1. Hence it is evident, that, in any *Right-angled Triangle*, a *Perpendicular*, being let fall from the *Right-angle* upon the *Hypothenuse*, will be a *Mean* proportional between the *Segments* of the *Hypothenuse:* That is, $BP : PA :: PA : PC$.

2. The Base (BA) is a *Mean* proportional between the *Hypothenuse* (BC) and that *Segment* of the *Hypothenuse* next to the Base, (*viz.* BP) that is, $BC : BA :: BA : BP$.

3. The *Cathetus* (AC) is a *Mean* proportional between the *Hypothenuse* (BC) and that *Segment* of the *Hypothenuse* next to the *Cathetus* (*viz.* PC): That is, $BC : AC :: AC : PC$.

Scholium.

I have been more large upon this most excellent *Theorem*, in giving a double *Demonstration* of it, because it is so *universally useful* in all Parts of the *Mathematics:* For the Business of *Trigonometry* (*both Plain and Spherical*) wholly depends upon it; and therefore one may truly say, that *Astronomy*, *Dialing*, *Navigation*, *Surveying*, *Optics*, &c. depend upon a due Application of it.

And of its Use in *Geometry*, *Des Cartes* takes particular Notice; as you may find in Dr. *Pell*'s Algebra, *Page* 65, whose Words are these:

Des Cartes, in a Letter not yet printed, writes thus: " In " searching the *Solution* of *Geometrical Questions*, I always make " use of Lines parallel and perpendicular, as much as is possible; " [*he means as many Lines as are useful*] and I consider no " other Theorems but these two, [*the Sides of like* Trian- " gles *have like Proportion*]. And [*in Rectangle Triangles* " *the*

" *the Square of the greatest Side is equal to the Squares of the* " *two other Sides.*] And I am not afraid to suppose many un- " known Quantities, that I may reduce the proposed Question " to such Terms, as to depend on no other Theorems but these " Two."

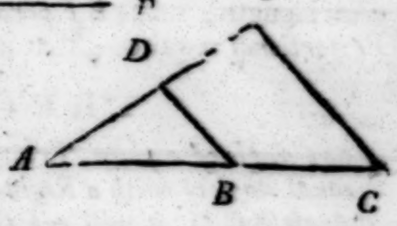
This I thought convenient to insert, that the young *Learner* may see how the great *Des Cartes* esteemed these two Theorems, *viz.* the last, and Theorem 11; for, in Truth, all the precedent Theorems are only (as it were) Preparatives to these Two.

This last Theorem demonstrates the *Reason* of the *Method* used in finding out *Proportional Lines*; as in the three following *Problems*.

PROBLEM I.

Two Right-lines being given, to find a Third in Proportion to them. (11. *e.* 6.)

Let these two *Lines* be $\begin{cases} A \text{———} B \\ A \text{———} D \end{cases}$

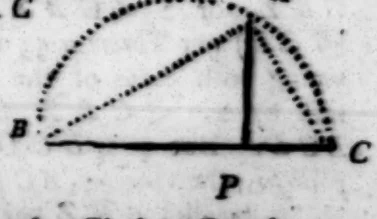


Set the Two given *Lines* at any Angle in the Point *A*, and produce the Line *AB* to *C*, making $BC = AD$; join the Points *BD* with a *Right-line*, and draw *CF* parallel to *BD*; then will the $\triangle\ ABD$ be like the $\triangle\ ACF$. Therefore $AB : BC\ (=AD) :: AD : DF$, which is the third *Proportional* required.

PROBLEM II.

Two Right-lines being given, to find a Mean proportional Line between them. (13, *e.* 6.)

Let the given *Lines* be $\begin{cases} B \text{———} P \\ P \text{———} C \end{cases}$



Join the two given *Lines* into one, *viz.* make $BC = BP + PC$, and upon *BC*, as *Diameter*, describe a *Semicircle*; then upon the Point *P*, where the two *Lines meet*, erect a Perpendicular to touch the Circle's *Periphery*, as

PA, and it will be the *Mean proportional* required, viz. $BP : AP :: AP : PC$.

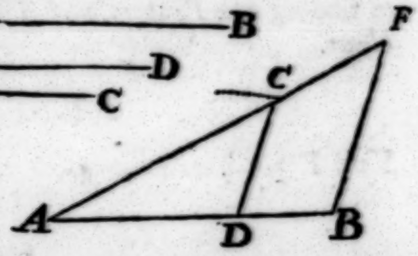
By this *Problem* 'tis easy to conceive how to make a *Square* equal to any given *Parallelogram*. (14. *e.* 6.)

For if *BP* be the *Length*, and *PC* be the *Breadth* of the given *Parallelogram*, then will *AB* be the Side of the *Square*, equal in *Area* to that *Parallelogram*.

PROBLEM III.

Three Right-lines being given, to find a fourth Proportional Line. (12. *e.* 6.)

Suppose the three Lines $\begin{cases} A \text{———} B \\ A \text{———} D \\ D \text{———} C \end{cases}$

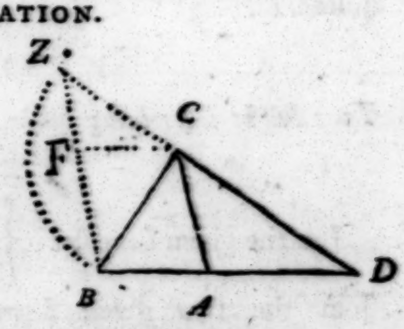


Upon the longest Line *AB* set off the next longest Line *AD*; *viz.* make $DB = AB - AD$; then upon the Point *D* set the other Line *DC* at an *Angle*, either *right* or *oblique*, and draw the *Right-line AC* continuing it a sufficient Length; make *BF parallel* to *DC*, and it will be the fourth *Proportional* required; that is $AD : DC :: AB : BF$.

THEOREM XIV.

If any Angle of a plain Triangle be bisected (i. e. *divided into two equal Angles) with a Right-line,* (viz. *as CA is supposed to do the Angle BCD) it will cut the opposite Side* (viz. *BD) in Proportion to the other two Sides of the Triangle.* (3. *e.* 6.) i. e. $BA : BC :: AD : CD$.

Demonstration.



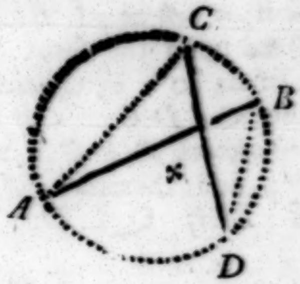
Produce the Side *DC*, until $CZ = CB$: join the Points *ZB* with a *Right-line*, and draw the Line *FC* parallel to *BD*; whence the $\angle Z = \angle CBZ$; per *Theorem* 6. and $\angle Z + \angle CBZ$, or $2 \angle CBZ = \angle (BCD)$ per *Theorem* 5; or, dividing both Sides of the Equation by 2, $\angle CBZ = \frac{1}{2} \angle BCD$. But $\frac{1}{2} \angle BCD = \angle ACB = \angle ACD$ by the *Hypothesis*, therefore $\angle ACB = \angle CBZ$ per *Axiom* 5: Whence *AC* is parallel to *BZ* per *Theorem* 3. and the Triangles *BDZ*, *ADC*, and *FCZ* are similar by the second Figure to *Theorem* 12. consequently $BA (= FC) : BC (= ZC) :: AD : CD$. Q.E.D.

THEO-

THEOREM XV.

If two Right-lines (howsoever drawn) within a Circle *do cut each other, the Rectangle made of the Segments (or Parts) of the one Line, will be equal to the Rectangle made of the Segments (or Parts) of the other Line.* (35. *e.* 3.)

That is, if two Lines (as AB and CD) do cut each other in any Point, as at x, then will $Ax \times Bx = Dx \times Cx$.



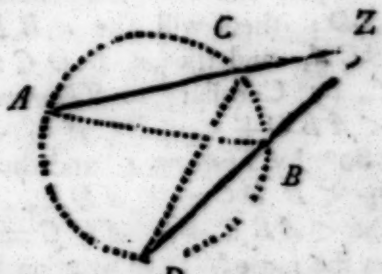
Demonstration.

Join the Points AC and BD with *Right-lines*, then will the $\triangle\ CxA$ be like to $\triangle\ BxD$: For $< B = < C$ and $< A = < D$, by Corollary to *Theorem* 9, and $< AxC = < BxD$, by *Theorem* 2. Therefore it will be $Ax : Dx :: Cx : Bx$, by *Theorem* 13. Consequently $Ax \times Bx = Dx \times Cx$. Q. E. D.

THEOREM XVI.

If two Right-lines are so drawn within a Circle, *as, being continued, they will meet in a Point out of the* Circle's *Periphery, the Rectangle made of the one whole Line, and its Part out of the* Circle, *will be equal to the Rectangle of the other whole Line, and its Part out of the* Circle. (36, 37. *e.* 3.)

That is, if the Lines AC and DB be continued unto the Point Z; then will $AZ \times CZ = DZ \times BZ$.



Demonstration.

Draw the Lines AB and CD, then will $\triangle\ CZD$ be like to the $\triangle\ BZA$; for $< A = < D$, and $< Z$ is common to both *Triangles*, consequently, $< ABZ = < DCZ$, by *Theorem* 4: therefore $AZ : BZ :: DZ :: CZ$. *Ergo*, $AZ \times CZ = DZ \times BZ$.

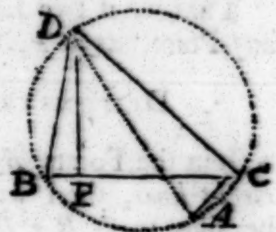
THEOREM XVII.

If from any Angle of a plain Triangle inscribed in a Circle *there be let fall a Perpendicular upon the opposite Side, as* DP;

 As

As that Perpendicular is in Proportion to one of the Sides including the Angle, so is the other Side including the Angle to the Diameter of the Circle.

DEMONSTRATION.



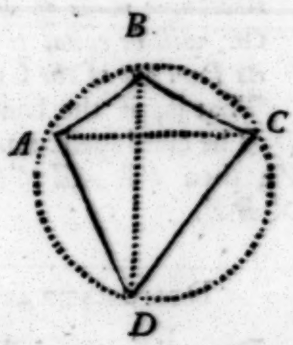
Let BCD be the proposed *Triangle*. From the $<$ at D draw the Diameter DA; complete the *Triangle* DCA then will $<A = <B$, because they both stand upon the same Arch DC, and $<DCA = 90°$, by *Theorem* 10. consequently the $<ADC = <BDP$ by *Theorem* 4. Therefore $\triangle DCA$ is like to the $\triangle DPB$; and therefore, $PD : DB :: DC : DA$: or $DP : DC :: DB : DA$, Q. E. D.

THEOREM XVIII.

If a Quadrangle (that is, a Trapezium) *be inscribed within a* Circle, *the two opposite Angles, taken together, are equal to two Right Angles, viz.* 180° (22 *e.* 3.)

That is, in the *Quadrangle* $ABCD$ the $<A + <C = 180°$. And the $<B + <D = 180°$.

DEMONSTRATION.



Draw the *two* Diagonals AC and BD; then will the $<BDA = <BCA$, and the $<BDC = <BAC$, by Corollary to *Theorem* 9. But $<ABC + <<BCA + <BAC = 180°$. by *Theorem* 4. and the $<BDA + <BDC = <ADC$. Therefore the $<ABC + <ADC = 180°$, and by the same Way of arguing it may be proved, that the $<BAD + <BCD = 180°$. Q. E. D.

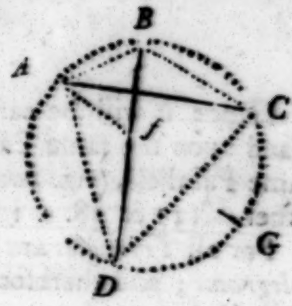
THEOREM XIX.

If in any Quadrangle inscribed within a Circle *there be drawn two Diagonals, as* AC *and* BD, *the Rectangle made of the two Diagonals will be equal to both the Rectangles made of the opposite Sides of the Quadrangle.*

That is, $AC \times BD = AB \times CD + AD \times BC$.

DEMON-

DEMONSTRATION.



Make the Arch $DG =$ Arch BC, and from the Points A, G draw the Line Af, and it will form the $\triangle AfD$, like to the $\triangle ABC$: For the $< fAD = < BAC$, because the Arches DG and BC are equal.

Again, the $< fDA = < BCA$, because they both stand upon the Arch AB: Consequently the $< AfD = < ABC$, by *Theorem* 4. Therefore it will be $AC : BC :: AD : Df$, by *Theorem* 13. *Ergo* $\frac{BC \times AD}{AC} = Df$.

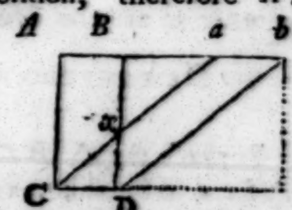
Again, the $\triangle BAf$ and $\triangle ACD$ are alike: For $< ABf = < ACD$, and $< BAf = < CAD$, because the $< fAD = < BAC$, and the $< CAf$ is common to both *Triangles*. Consequently the $< AfB = < ADC$. Therefore $AC : CD :: AB : Bf$, by *Theorem* 13. *Ergo* $\frac{CD \times AB}{AC} = Bf$. But $Df + Bf = BD$. Consequently, $BC \times AD + CD \times AB = BD \times AC$. Q.E.D.

THEOREM XX.

All Parallelograms (whether Right or Oblique angled) that stand upon the same Base, *or upon equal* Bases, *and betwixt the same Parallels, are equal to one another.* (35 & 36. *e.* 1.)

That is, $\square ABCD = \square abCD$.

DEMONSTRATION.



Because $AB = CD = ab$, by Supposition, therefore $Aa = Bb$; for Ba is common to both. And because $AC = BD$, and the $< A = < B$, therefore the $\triangle ACa = \triangle BDb$: And if from both Triangles there be taken the $\triangle Bxa$ common to both, there will remain the *Trapezium* $ABxC = abxD$, per *Axiom* 5.

But

But the *Trapezium* $ABxC + \triangle CxD = \square ABCD$, and the *Trapezium* $abxD + \triangle CxD = \square abCD$, ccnfequently, $\square ABCD = \square abCD$. Q. E. D.

Corollary.

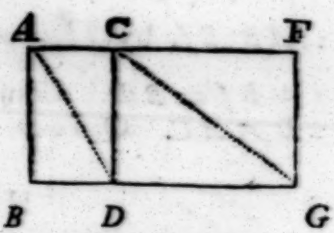
Hence it will be eafy to conceive, that all *Triangles* which ftand upon the fame Bafe, or upon equal Bafes, and between the fame Parallels, (viz. *having the fame Height)* are equal one to another. (37 & 38. *e.* 1.)

For all *Triangles* are the Halves of their circumfcribing *Parallelograms*; and therefore, if the Wholes be equal, their Halves will alfo be equal.

THEOREM XXI.

Parallelograms *(and confequently* Triangles) *which have the fame Heighth, have the fame Proportion one to another as their* Bafes *have.* (1. *e.* 6.)

Demonstration.



Draw AF parallel to BG, and draw AB, CD, FG Perpendiculars to them. Then will $BD \times AB = \square ABCD$. And becaufe $CD = AB$, therefore $DG \times AB = \square CDFG$, but $BD : DG :: BD \times AB : DG \times AB$. And confequently $\triangle ABD : \triangle CDG :: BD : DG$, &c.

Q. E. D,

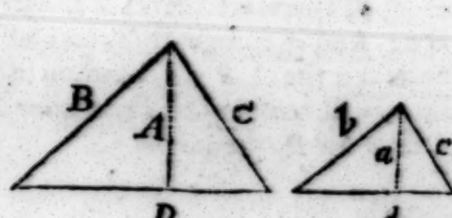
THEOREM XXII.

Like Triangles are in a duplicate Ratio *to that of their homologous Sides.* (19. *e.* 6.)

That is, the *Area*'s of like *Triangles* are in *Proportion* one to another as are the *Squares* of their like Sides.

Demonstration.

Suppofe the $\triangle BCD$ and $\triangle bcd$ to be alike, and their like Sides to be thofe marked with the fame Letters.



Let

Let A and a be Perpendiculars to the two Bases D and d.

$$\left.\begin{array}{l}\text{The } \tfrac{1}{2}\,DA = \text{the Area of } \triangle\, BCD \\ \text{And } \tfrac{1}{2}\,da = \text{the Area of } \triangle\, bcd\end{array}\right\}$$ By *Lemma* 3, *Page* 303.

But	1	$\left.\begin{array}{l} B:b::D:d \\ B:b::A:a \end{array}\right\}$ &c. By *Theorem* 13.
And	2	
Conseq.	3	$D:d::A:a$
$3 \therefore$	4	$Da = dA$
$4 \times \frac{1}{2} Dd$	5	$\frac{1}{2} DDda = \frac{1}{2} DddA$. By *Axiom* 3.
5, Hence	6	$DD:dd::\frac{1}{2}DA:\frac{1}{2}da$. And so for other Sides.

Q. E D.

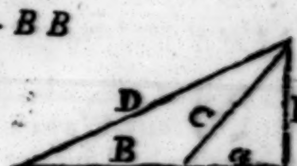
THEOREM XXIII.

In every Obtuse-angled Triangle (as BCD) the Square of the Side subtending the obtuse Angle (as D) is greater than the Squares of the other two Sides (B and C) by a double Rectangle made out of one of the Sides (as B) and the Segment or Part of that Side produced, (as a) until it meet with the Perpendicular (P) let fall upon it. (12. e. 2.)

That is, $DD = BB + CC + 2\,Ba$.

Demonstration.

First	1	$DD = PP + aa + 2\,Ba + BB$
And	2	$CC = PP + aa$
$1 - 2$	3	$DD - CC = 2Ba + BB$
$1 + CC$	4	$DD = BB + CC + 2Ba$



Q. E. D.

Corollary.

Hence it is evident, that, if the Sides of any *Obtuse-angled Triangle* are given, the *Segment* (a) of the Side produced (or the Perpendicular P) may be easily found.

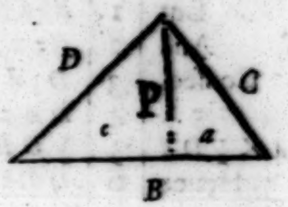
THEOREM XXIV.

If a Perpendicular *(as P) be let fall into any Acute-angled Triangle (as BCD), the Square of either of the two Sides (as D) is less than the Squares of the other Side, and that Side upon which the* Perpendicular *falls* (viz. *C and B*) *by a double Rectangle made of the Side B, and that Segment or Part of it* (viz. *a*) *which lies next to the Side C.* (13. *e.* 2.) That is, $DD + 2\,Ba = BB + CC$.

DEMON-

Demonstration.

First	1	$DD = PP + ee$	By *Theo.* 11.
And	2	$CC = PP + aa$	
But	3	$B - a = e$, by Figure.	
3 ⊙ 2	4	$BB - 2Ba + aa = ee$.	
4 — aa	5	$BB - 2Ba = ee - aa$.	
1 — 2	6	$DD - CC = ee - aa$.	
5, 6	7	$DD - CC = BB - 2Ba$.	
7 +	8	$DD + 2Ba = BB + CC$.	



Q. E. D

Corollary.

Hence it follows, that, if the Sides of any *Acute-angled Triangle* be known, the Perpendicular *P.* and the Segments of the Side whereon it falls (*viz.* *a*, *e.*) may be easily found.

CHAP. IV.

The SOLUTION *of several Easy* PROBLEMS *in plain Geometry, whereby the Learner may (in Part) perceive the Application or Use of the foregoing Theorems.*

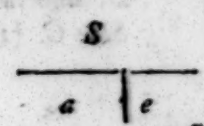
" NOTE, when a Line, or the Side of any plain Triangle, is " any Way cut into two or more Parts, either by a a Per- " pendicular Line let fall upon it, or otherwise, those Parts are " usually called *Segments*; and so much as one of those Parts is " longer than the other, is called the *Difference of the Segments.*

" And when any Side of a Triangle, or any Segment of its Side " is given, it is usually marked with a small Line cross it, thus: " ——|—— and those Sides or Parts of Sides, that are sought, " are marked with four Points, thus — : : —

PROBLEM I.

To cut or divide a given Right-line (as S) into Extreme and Mean Proportion. (11. *e.* 2.)

That is, to *divide a Line* so, that the *Square* of the *greater Segment* (or *Part*) *a*, may be equal to the *Rectangle* made of the whole Line *S*, and the lesser Segment *e*.

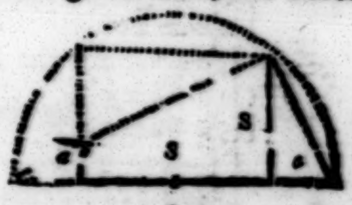


Viz.	1	$Se = aa$, by the *Problem.*
And	2	$S - a = e$, for $S = a + e$.

$1 \div S$	3	$\frac{aa}{S}=e$
2 *and* 3	4	$\frac{aa}{S}=S-a$. By *Axiom* 5.
$4 \times S$	5	$aa=SS-Sa$
$5+Sa$	6	$aa+Sa=SS$
6, solved	7	$a=\sqrt{SS+\frac{1}{4}SS}:-\frac{1}{2}SS$. See *Pages* 195, 196.

Note, *The last* Problem *cannot be truly answered by Numbers, but Geometrically it may be performed, thus:*

1. Take a Square, whose Side is $=S$ the given Line, and bisect one of its Sides in the Middle, as at C; upon the Point C describe such a Semicircle as will pass thro' the remotest Points of the Square, and compleat its Diameter.



C

2. Then will either Part of the Diameter, on each End of the Side S, be $=a$, the *greater Segment* sought.

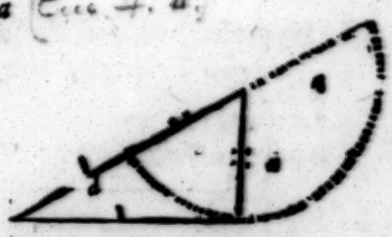
But $a+S:S::S:a$. By *Theorem* 13.

Ergo, $aa+Sa=SS$. Which was to be done.

PROBLEM II.

The Base *of any Right-angled Triangle, and the Difference between the* Hypothenuse *and* Cathetus *being given, to find the* Cathetus, *&c.*

Let {	1	$b=72$
	2	$d=32$
And	3	$a=$ *Cathetus* sought
Then	4	$bb+aa=dd+2da+aa$ By *Theorem* 11.
$4-aa$	5	$bb=dd+2da$
$5-dd$	6	$2da=bb-dd$
$6 \div 2d$	7	$a=\frac{bb-dd}{2d}=65$
Or,	8	$b:d+2a::d:b$. By *Theorem* 13.
8 ∵	9	$bb=dd+2da$. As before at the 5th Step.



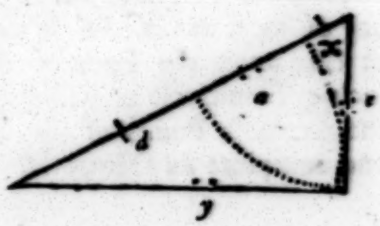
b

Here you see that either Way raises the same *Equation*; neither is there any constant *Method* or *Road* to be observed in solving *Geometrical Problems*, but every one makes Use of such Ways and Theorems as happen to come first into their Mind, the Result being every Way the same.

PROBLEM III.

The Difference between the Base *and* Hypothenuse *of any Right-angled Triangle, and the Difference between the* Cathetus *and* Hypothenuse *being both given, to find the Triangle.*

Let	1	$d = 32$
	2	$x = 25$
And	3	$d+x+a=$ the *Hypot.*
Then	4	$d+a=y$ } by the *Probl.*
	5	$x+a=e$
4 ☉ 2	6	$dd+2da+aa=yy$
5 ☉ 2	7	$xx+2xa+aa=ee$
3 ☉ 2	8	$dd+2dx+2da+2xa+xx+aa=\square$ *Hypothenuse.*
6 + 7	9	$dd+2da+2xa+xx+2aa=yy+ee.$



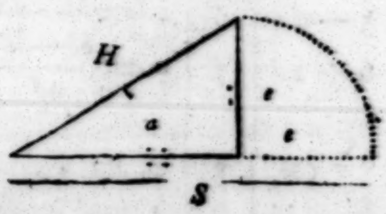
These two last Steps are equal, by *Theorem* 11. Consequently, if those Things that are equal in both be taken away, the Remainders will be equal. By *Axiom* 2.

That is	10	$aa = 2dx = 1600$
10 ωω 2	11	$a = \sqrt{2dx} = 40$
1 + 11	12	$d + a = 72 = y$ *The* Base.
2 + 11	13	$x + a = 65 = e$ *The* Cathetus.
1+2+11	14	$d + x + a = 97$ *The* Hypothenuse.

PROBLEM IV.

The Hypothenuse, *and the Sum of the other two Sides, of any Right-angled Triangle, being given, thence to find the Sides.*

Let	1	$H = 97$
And	2	$a+e=S=137$
By *Fig.*	3	$aa+ee=HH$
2 ☉ 2	4	$aa+2ae+ee=SS$
4 — 3	5	$2ae=SS-HH$
3 — 5	6	$aa-2ae+ee=2HH-SS$
6 ωω 2	7	$a-e=\sqrt{2HH-SS}$



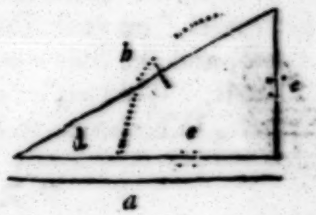
2 + 7

$2+7$	8	$2a = S + \sqrt{2HH - SS} = 144$
$8 \div 2$	9	$a = \frac{S + \sqrt{2HH - SS}}{2} = 72$ *The* Base *required.*
$2 - 9$	10	$e = \frac{S - \sqrt{2HH - SS}}{2} = 65$ *The* Cathetus.

PROBLEM V.

The Hypothenuse, *and the Difference of the other two Sides of any Right-angled Triangle being given, to find the Sides.*

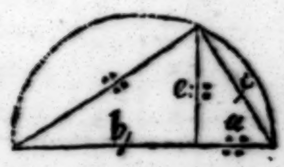
Let	1	$b = 97$ As before.
And	2	$a - e = d = 7$ Quære a
By *Fig.*	3	$aa + ee = bb$
$2\ \odot^2$	4	$aa - 2ae + ee = dd$
$3 - 4$	5	$2ae = bb - dd$
$3 + 5$	6	$aa + 2ae + ee = 2bb - dd$
$6\ \omega^2$	7	$a + e = \sqrt{2bb - dd}$
$2 + 7$	8	$2a = d + \sqrt{2bb - dd} = 144$
$8 \div 2$	9	$a = 72$
$7 - 2$	10	$2e = \sqrt{2bb - dd} : - d = 130$
$1 \div 2$	11	$e = 65$



PROBLEM VI.

In any Right-angled Triangle, either the Base, *or* Cathetus, *and the alternate Segment of the* Hypothenuse *made by a Perpendicular let fall from the Right-angle, being given, to find the other Segment.*

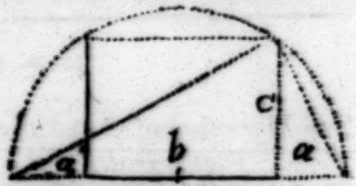
Let	1	$c = 45$ *The* Cathetus
And	2	$b = 48$ *The* alternate *Segm.*
Then	3	$b : e :: e : a$ Quære a
$3\ \because$	4	$ba = ee$
Again,	5	$cc - aa = ee$. By *Theor.* 11.
$4, 5$	6	$ba = cc - aa$
$6 + aa$	7	$aa + ba = cc$
$7, C\square$	8	$aa + ba + \frac{1}{4}bb = cc + \frac{1}{4}bb$
$8\ \omega^2$	9	$a + \frac{1}{2}b = \sqrt{cc + \frac{1}{4}bb}$
$9 - \frac{1}{2}b$	10	$a = \sqrt{cc + \frac{1}{4}bb} : \frac{1}{2}b = 27$ And so on for e, &c.



I shall

I shall now shew the *Geometrical Construction* (or *Solution*) of the three Cases of *Quadratic Equations* promised in page 202. Let the first Example be that above, *viz.* $aa + ba = cc$. Case 1.

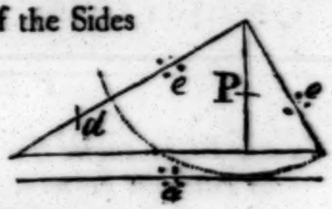
Make the Co-efficient b, and the *Root* of the *Resolvend* (which is here) c, into a *Right-angled Parallelogram.* And upon the middle Point of the Side $= b$ describe such a Semicircle, as will pass thro' the remotest Points or Angles of the *Parallelogram*, compleating its Diameter, as in the annexed *Scheme.*



Then will either Part of the Diameter, on each End, be equal to a; the other Part will be $a + b$, and the Side c will be a *mean Proportional* between them: That is, $a + b : c :: c : a$. By *Theorem* 13, consequently $aa + ba = cc$. Which was to be done.

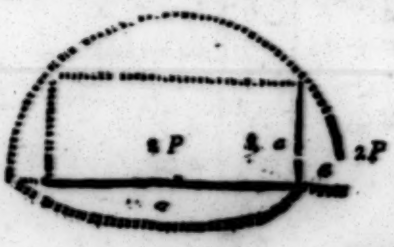
PROBLEM VII.

The Difference between the Base *and* Cathetus *of any Right-angled Triangle, and the* Perpendicular *let fall from the Right-angle upon the* Hypothenuse, *being given; thence to find the* Hypothenuse, *&c.*



Let	1	$d=15$ The Difference of the Sides
And	2	$p=36$
Quere a	3	$a=$ *The* Hypothenuse.
By *Fig.*	4	$d+e:p::a:e$
4 ∵	5	$de+ee=pa$
Again,	6	$dd+2de+2ee=aa$. By *Theorem* 11.
5×2	7	$2de+2ee=2pa$
$6 - 7$	8	$dd=aa-2pa$. Case 2.
8 C □	9	$aa-2pa+pp=dd+pp=1521$.
$9\ \mathrm{ww}^2$	10	$a-p=\sqrt{dd+pp}=39$
$10+p$	11	$a=p+\sqrt{dd+pp}=75$, &c. for e per *Step* 5.

The Geometrical Construction of this Case 2, viz. $aa-2pa=dd$ may be performed in the very same Manner as the *last Case* was; that is, by making a Right angled Parallelogram of the Co-efficient $2p$ and the $\sqrt{dd}$, viz. d, &c. As in the annexed Figure.

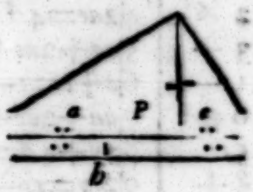


Then

Then will the greater Part of the Diameter to one End of the *Parallelogram* be $=a$, and the lesser Part will be $a-2p$; For $a:d::d:a-2p$ by *Theorem* 13. Consequently, $aa-2pa=dd$. Which was to be done.

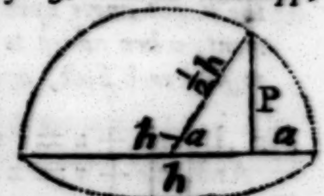
PROBLEM VIII.

The Hypothenuse *of any Right-angled Triangle, and the* Perpendicular *let fall from the Right-angle upon the* Hypothenuse, *being given, to find the greater Segment of the* Hypothenuse, &c.



Let	1	$b=75$ *The* Hypothenuse
And	2	$p=36$
Then	3	$a+e=b$ Quære a
per *Fig.*	4	$a:p::p:e$
4 ∵	5	$\frac{pp}{a}=e$
$3-a$	6	$b-a=e$
5, 6	7	$b-a=\frac{pp}{a}$
$7\times a$	8	$ba-aa=pp$ Case 3.
$8\mp$	9	$aa-ba=-pp$
9 C□	10	$aa-ba+\frac{1}{4}bb=\overline{\frac{1}{4}bb+pp}=110,25$
10 ⍵⍵	11	$a-\frac{1}{2}b=\sqrt{\frac{1}{4}bb-pp}=10,5$
$11+\frac{1}{2}b$	12	$a=b\frac{1}{4}+\sqrt{\frac{1}{2}bb-pp}=48$. Or, $a=27$.

The Geometrical Construction of *Case* 3, viz. $ba-aa=pp$, may be thus performed: Draw a *Right-line* (of any convenient Length at Pleasure) and near its Middle erect a Perpendicular $=p$, *viz.* of the same Length with the *Root* of the *Resolvend.* From the top Point or upper End of that Perpendicular, set off half the Length of the Co-efficient, *viz.* $\frac{b}{2}$ and upon the Point where $\frac{b}{2}$ just touches the first Line (with the same Distance) describe a Semicircle; then will its Diameter b be cut by the Perpendicular p into two Segments, which are the two Values of the Root a, *viz.* the *greater* and *lesser Roots*, both taken together, being always equal to the Co-efficient: (*vide* Page 201.) For $b-a:p::p:a$ by *Theorem* 13. Ergo, $ba-aa=pp$. Which was to be done.



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PROBLEM IX.

The Perimeter, *i. e. the Sum of all the three Sides of any Right-angled Triangle, and its* Area, *being given, thence to find each Side.*

Viz. Let	1	$a+e+y=s=234$ The *Sum* of the *Sides*.
And	2	$ae=A$ The *Area*$=2340$
Again	3	$aa+ee=yy$ By Figure
2×4	4	$2ae=4A$
$3+4$	5	$aa+2ae+ee=yy+4A$
$1-y$	6	$a+e=s-y$
6 ⊙^{2}	7	$aa+2ae+ee=ss-2sy+yy$
5, 7	8	$yy+4A=ss-2sy+yy$
$8\pm$	9	$2sy=ss-4A=45396$
$9\div 2s$	10	$y=\frac{ss-4A}{2s}=\frac{1}{2}s-\frac{2A}{s}=97$ *The* Hypothenuse.
6, 10	11	$a+e=s-y=137$
$3-4$	12	$aa-2ae+ee=yy-4A=49$
12 $\sqrt{}$	13	$a-e=\surd 46=7$
$11+13$	14	$2a=137+7=144$
$13\div 2$	15	$a=72$ *The* Base.
$11-15$	16	$e=137-72=65$ *The* Cathetus.

PROBLEM X.

In any Right-angled Triangle a Perpendicular *being let fall from the Right-angle upon the* Hypothenuse; *if the Sum of each Segment, when added to its adjacent or next Side, be given, thence to find each Side, and the Segments.*

Viz. If	1	$a+u=s=108$
And	2	$e+y=z=72$
To find		$a, e, u, y,$ and dp
$1-a$	3	$u=s-a$
3 ⊙^{2}	4	$uu=ss-2sa+aa$
$4-aa$	5	$uu-aa=ss-2sa=pp$
$2-e$	6	$z-e=y$
6 ⊙^{2}	7	$zz-2ze+ee=yy$
$7-ee$	8	$zz-2ze=yy-ee=pp$
5, 8	9	$zz-2ze=ss-2sa$
By *Fig.*	10	$a:p::p:e$
10 ∵	11	$ae=pp$
5, 11	12	$ae=ss-2sa$

$12\div a$

$12 \div a$	13	$e = \dfrac{ss - 2sa}{a}$
$13 \times 2z$	14	$2ze = \dfrac{2zss - 4zsa}{a}$
$9 + 14$	15	$zz = ss - 2sa + \dfrac{2zss + 4zsa}{a}$
$15 \times a$	16	$zza = ssa - 2saa + 2zss - 4zsa$
$16 \pm$	17	$2saa + zza + 4zsa - ssa = 2zss$
$17 \div 2s$	18	$aa + \dfrac{zza}{2s} + 2za - \frac{1}{2}sa = zs$
Substitute	19	$2x = \dfrac{zz}{2s} + 2z - \frac{1}{2}s = 114$
Then	20	$aa + 2xa = zs = 7776$
20 C □	21	$aa + 2xa + xx = zs + xx = 11025$
21 vw^2	22	$a + x = \sqrt{zs + xx} = 105$
$22 - x$	23	$a = \sqrt{zs + xx} : - x = 48$
$1 - 23$	24	$a = 60 =$ *The* Base.
per 13	25	$e = \dfrac{ss}{a} - 2s = 27$
$2 - 25$	26	$y = 45 =$ *the* Cathetus.
$23 + 25$	27	$a + e = 75 =$ *the* Hypothenuse.

PROBLEM XI.

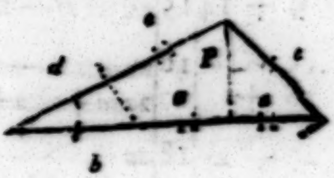
The Difference of the Sides of any Oblique-angled plain Triangle, the Difference of the Segments of the Base, *and the Difference between the greater Side and the* Base, *being given, to find the* Base, *&c.*

Let	1	$d =$ the Difference of the Sides $= 405$
	2	$b =$ the Difference of the Segments $= 495$
	3	$x = 165$ the Difference of the greater Side and Base
And	4	$a =$ the least Side
Then	5	$d + a + x =$ the Base
And	6	$d + a + x : d + 2a :: n : b$
		By *Theorem* 16.
6 ∵	7	$db + ba + bx = dd + 2da$
$7 \pm$	8	$2da - ba = db + bx - dd$
$8 \div 2d - b$	9	$a = \dfrac{db + bx - dd}{2d - b} = \dfrac{118125}{315} = 375$
$1 + 9$	10	$d + a = 780 =$ *the* greatest Side.
$3 + 10$	11	$d + a + x = 945 =$ *the* Base.

PRO-

PROBLEM XII.

The Difference of the Sides of any plain Triangle, the Difference of the Segments of the Base, *and the* Perpendicular *let fall from the vertical Angle, being given, thence to find all the Sides.*



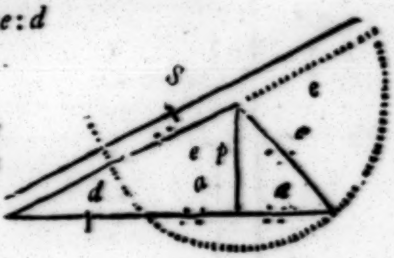
Let	1	$d=405$ } as before.
	2	$b=495$
And	3	$p=300$
Quære	4	$a=$ the lesser *Segment.*
Then	5	$b+2a : d+2e :: d : b.$
5 ∵	6	$bb+2ba=dd+2de$
6 — dd	7	$bb-dd+2ba=2de$
Substitute	8	$2x=bb-dd=81000$
7, 8	9	$2x+2ba=2de$
9 ÷ 2d	10	$\frac{x+ba}{d}=e$
But	11	$pp+aa=ee$ By *Theorem* 11.
10 ⊙²	12	$\frac{xx+2xba+bbaa}{dd}=ee$
11, 12	13	$\frac{xx+2xba+bbaa}{dd}=pp+aa$
13 × dd	14	$xx+2xba+bbaa=ppdd+ddaa$
14 ±	15	$bbaa-ddaa+2xba=ppdd-xx$
8, 15	16	$2xaa+2xba=ppdd-xx$
16 ÷ 2x	17	$aa+ba=\frac{ppdd}{2x}-\frac{1}{2}x$
17 C □	18	$aa+ba+\frac{1}{4}bb=\frac{1}{4}bb \div \frac{ppdd}{2x}-\frac{1}{2}x$
18 ꟿ²	19	$a+\frac{1}{2}b+\sqrt{\frac{1}{4}bb+\frac{ppdd}{2x}-\frac{1}{2}x}$
19 — ½b	20	$a=\sqrt{\frac{1}{4}bb+\frac{ppdd}{2x}-\frac{1}{2}x} : -\frac{1}{2}b=225$
20 × 2	21	$2a=450$
2 + 21	22	$b+2a=945$ *the* Base
10, *Num.*	23	$e=375=$ *the* lesser Side.
1 + 23	24	$d+e=780=$ *the* greater Side.

PROBLEM XIII.

The Sum of the two Sides of any plain Triangle, the Difference of the Segments of the Base, *and the* Perpendicular *let fall from the Vertical*

Vertical Angle upon the Base, *being given, thence to find the* Base *and the Sides.*

Let	1	$s=1155$ the *Sum* of the *Sides.*
	2	$d=495$ the *Difference* of the *Segments.*
	3	$p=300$ the *Perpendicular.*
Put	4	$a=$ the least *Segment.*
	5	$e=$ the least *Side.*
Then	6	$d+2a=$ the *Base.*
And	7	$s-2e=$ the *Difference* of the *Sides.*
Per *Fig.*	8	$d+2a:s::s-2e:d$
	9	$aa+pp=ee$
9 ɯ^2	10	$\sqrt{aa+pp}=ee$
8 ∵	11	$dd+2da=ss-2se$
11 $\pm$	12	$2se=ss-dd-2da$
Suppose	13	$2x=ss-dd$
Then	14	$2se=2x-2da$
14 $\div$ 2s	15	$e=\frac{x-da}{s}$
10, 15	16	$\frac{x-da}{s}=\sqrt{aa+pp}$
16 ⊙^{2}	17	$\frac{xx-2xda-ddaa}{ss}=aa+pp$
17 × ss	18	$xx-2xda+ddaa=ssaa+sspp$
18 $\pm$	19	$ssaa-ddaa+2xda=xx-sspp$
13 19	20	$2xaa+2xda=xx-sspp$
20 $\div$ 2x	21	$aa+da=\frac{1}{2}x-\frac{sspp}{2x}$, &c. as before.
21, hence	22	$a=225$
22 × 2	23	$2a=450$
2 + 23	24	$d+2a=945$ the *Base.*
10, *Num.*	25	$e=375$ the lesser Side.
1 — 25	26	$s-e=780$ the greater Side.

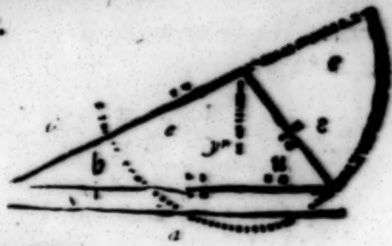


PROBLEM XIV.

The Area of any Oblique-angled plain Triangle, *the Difference of the Sides, and the Difference of the Segments of the* Base, *being given, thence to find the* Base, &c.

Let	1	$A=141750=$ the *Area.*
	2	$d=405$
	3	$b=495$

Put



Put	4	$y =$ the *Perpendicular*.
	5	$a =$ the *Base*.
Then	6	$\frac{1}{2} ya = A$
Per *Fig.*	7	$a : d + 2e :: d : b$
$7 \because$	8	$ba = dd + 2de$
$8 - dd$	9	$ba - dd = 2de$
$9 \odot^{2}$	10	$bbaa - 2ddba + dddd = 4ddee$
Per *Fig.*	11	$\frac{a-b}{2} = u$ the lesser Segment of the *Base*.
$11 \odot^{2}$	12	$\frac{aa - 2ba + bb}{4} = uu$
$6 \times \overline{2}$	13	$ya = 2A$
$13 \div a$	14	$y = \frac{2A}{a}$
$14 \odot^{2}$	15	$yy = \frac{4AA}{aa}$
Per *Fig.*	16	$yy + uu = ee = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
$10 \div 4dd$	17	$\frac{bbaa - 2ddba + dddd}{4dd} = ee$
16, 17	18	$\frac{bbaa - 2ddba + d^4}{4dd} = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
$18 \times aa$	19	$\frac{bba^4 - 2ddba^3 + d^4aa}{4dd} = 4AA + \frac{a^4 - 2ba^3 + bba^2}{4}$
$19 \times 4dd$	20	$\left\{ \begin{array}{l} bba^4 - 2ddba^3 + d^4aa = 16AAdd + dda^4 \\ - 2ddba^3 + ddbba^2 \end{array} \right.$
$20 \ \underline{\pm}$	21	$bba^4 + dda^4 + d^4a^2 - ddbba^2 = 16AAdd$
$21 \div$	22	$aaaa - dddaa = \frac{16AAdd}{bb - dd}$
22 C $\square$	23	$aaaa - ddaa + d\frac{1}{4}ddd = \frac{16AAdd}{bb - dd} + \frac{1}{4} d^4$
$23\ \mathrm{ɯ}^{2}$	24	$aa - \frac{1}{2}dd = \sqrt{\frac{16AAd}{bb - dd} + \frac{1}{4} d^4}$
$24 + \frac{1}{2}dd$	25	$aa = \frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4} d^4}$
$25\ \mathrm{ɯ}^{2}$	26	$a = \sqrt{:\frac{1}{2}dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4} d^4}} = 945$

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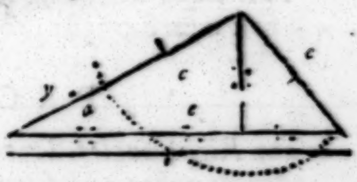
PROBLEM XV.

There is an Oblique-angled plain Triangle, wherein a Perpendicular *is let fall from the Vertical Angle upon the* Base; *the least Side and the Base are given; and the Rectangle of the Difference of the Sides into the least Side is equal to the Square of the Difference of the Segments of the* Base: *'Tis required to find the Segments of the* Base, *&c.*

Let	1	$c = 56 =$ the least Side.
	2	$B = 92 =$ the *Base*.
And	3	$a + 2c = B$.
Put	4	$y =$ the Difference of the Sides.
Then	5	$cy = aa$ by the Question.
By *Figure*	6	$B : 2c + y :: y : a$, for $B = a + 2c$
$6 \because$	7	$Ba = 2cy + yy$
5×2	8	$2cy = 2aa$
$7 - 8$	9	$Ba - 2aa = yy$
$5 \bullet^{2}$	10	$ccyy = aaaa$
$10 \div cc$	11	$yy = \frac{aaaa}{cc}$
9, 11	12	$Ba - 2aa = \frac{aaaa}{cc}$
$12 \times cc$	13	$ccBa - 2ccaa = aaaa$
$13 \div a$	14	$ccB - 2cca = aaa$
$14 + 2cca$	15	$aaa + 2cca = ccB$
15, in *Num.*	16	$aaa + 6272a = 288512$

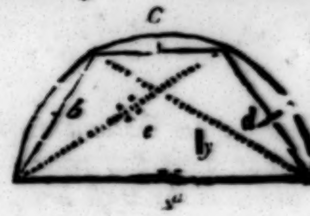


The Value of a, in this Equation, may be found as in the Examples Page 238, viz. by putting $r + e = a$, &c. as in those Examples you will find $a = 37,55502$, &c.

PROBLEM XVI.

The three Chords or Subtenses of three Arches completing a Semicircle *being each given, thence to find the Diameter of that* Circle, That is,

Any Trapezium being inscribed in a Semicircle, if one of its Sides be the Diameter, and the other three Sides be given, thence to find the Diameter or fourth Side.

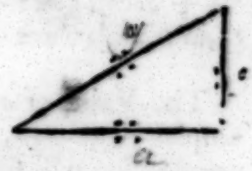


Let	1	$b=3$ } the 3 Sides.
	2	$c=4$
	3	$d=5$
Quere	4	$a=$ the *Diam.* sought
		Draw the two *Diagonals* e and y
Then	5	$ca+bd=ey$. By *Theorem* 19.
And	6	$aa-bb=yy$ } By *Theorem* 10 and 11.
	7	$aa-dd=ee$
$5 \circledcirc^2$	8	$ccaa+2bdca+bbdd=eeyy$
6×7	9	$aaaa-bbaa-ddaa+bbdd=eeyy$
$8,=9$	10	$aaaa-bbaa-ddaa=ccaa+2bdca$
$10 \div a$	11	$aaa-bba-dda=cca+2bdc$
$11-cca$	12	$aaa-bba-dda-cca=2bdc$
12, *Numb.*	13	$aaa-50a=120$

This Equation being solved as in Example 2, Page 240, you will find $a=8,05581$, &c.

PROBLEM XVII.

In any Right-angled Triangle, the Area *and the* Sum *of the* Hypothenuse *when added to either Side, being given, thence to find the Sides,* &c.



Suppose	1	$\frac{ae}{2}=A=1350$ the *Area.*
	2	$y+e=s=120$ the *Sum*, &c.
	3	Quære a, e, and y
$1\times\overline{2}$	4	$ae=2A$
$4\div a$	5	$e=\frac{2A}{a}$
Per *Fig.*	6	$aa+ee=yy$
$2-e$	7	$y=s-e$
5, 7	8	$y=s-\frac{2A}{a}$
$8 \circledcirc^2$	9	$yy=ss-\frac{4sA}{a}+\frac{4AA}{aa}$
$5 \circledcirc^2$	10	$ee=\frac{4AA}{aa}$
$10+aa$	11	$aa+ee=\frac{4AA}{aa}+aa$

6, 9, 11	12	$\frac{4AA}{aa}+aa=yy=ss-\frac{4sA}{a}+\frac{4AA}{aa}$
12, That is	13	$aa=ss-\frac{4sA}{a}$
$13\times a$	14	$aaa=ssa-4sA$
14 +	15	$ssa-aaa=4sA$
15, in *Num.*	16	$14400a-aaa=648000$

The Value of a, in this Equation, may be found as in the third Example, Page 241; that is, by making $r+e=a$, &c. it will be found that $a=60$.

PROBLEM XVIII.

There is an Oblique-angled plain Triangle, wherein a Perpendicular is let fall from the Verticle Angle *upon the Base; the Sum of each Segment of the Base, when added to its adjacent or next Side, and the* Area *of the* Triangle, *are given, to find the* Perpendicular *and each Side.*

Let	1	$y+b=z=1500$ — Quære y, b, e, and u
	2	$e+u=s=600$
	3	$A=$ the *Area* $=141750$
And	4	$a=$ the *Perpendicular* sought.
Then	5	$\overline{y+e}\times\frac{1}{2}a=A$
$5\times2\div a$	6	$y+e=\frac{2A}{a}$
Per *Fig.*	7	$yy+aa=bb$
	8	$ee+aa=uu$
$1-y$	9	$b=z-y$
$2-e$	10	$u=s-e$
$9\odot^{2}$	11	$bb=zz-2zy+yy$
$10\odot^{2}$	12	$uu=ss-2se+ee$
7, 11	13	$zz-2zy=aa$
8, 12	14	$ss-2se=aa$
13 +	15	$zz-aa=2zy$
14 +	16	$ss-aa=2se$
$15\div2z$	17	$\frac{zz-aa}{2z}=y$
$16\div2s$	18	$\frac{ss-aa}{2s}=e$
$17\div18$	19	$\frac{zz-aa}{2z}+\frac{ss-aa}{2s}=y+e$

Having found the Value of a from the 24th Step, e and y will be easily found by these two Steps, and b, u, by the 9th and 10th Step.

6, 19 | 20 | $\frac{zz - aa}{2z} + \frac{ss - aa}{2s} = \frac{2A}{a}$

23 × 2z | 21 | $zz - aa + \frac{zss - zaa}{s} = \frac{4zA}{a}$

21 × s | 22 | $zzs - saa + zss - zaa = \frac{4zA}{a}$

22 × a | 23 | $zzsa - saaa + zssa - zaaa = 4zAs$

23, *Numb.* | 24 | $900000a - aaa = 243000000$

Here $a = 300$ found as in the last *Problem.*

PROBLEM XIX.

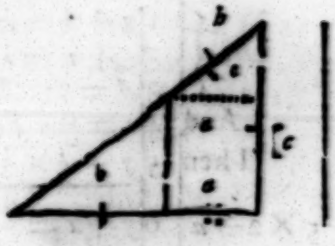
There is a Right-angled Triangle, wherein a Right-line is drawn parallel to the Cathetus; *there is given the* Cathetus, *that Segment of the* Hypothenuse *next to the* Cathetus, *and the alternate Segment of the* Base; *thence to find the* Base, &c.

viz. Let | 1 | $b = 20$. $c = 24$. and $h = 15$

Then | 2 | $b + a =$ the *Base*. Quære a

Here | 3 | $b + a : c :: a : e$ per *Figure.*

And | 4 | $aa + ee = hh$ per *Figure.*



3 ∵ | 5 | $\frac{ca}{b+a} = e$

5 ⊙² | 6 | $\frac{ccaa}{bb \div 2ba + aa} = ee$

4 — aa | 7 | $hh - aa = ee$

6, 7 | 8 | $\frac{ccaa}{bb + 2ba + aa} = hh - aa$

8 × | 9 | $ccaa = hhbb - bbaa + 2hhba - 8ba^3 + hhaa - a^4$

9 + | 10 | $a^4 + 2baaa + ccaa + bbaa - hhaa - 2hhba = hhbb$

That is, | 11 | $aaaa + 40aaa + 751aa - 9000a = 90000.$

For a *Solution* of this *Equation*, let it be made

$aaaa + baaa + caa - da = G$ *Viz.* $\begin{cases} b = 40 \ . \ c = 751 \\ d = 9000 \ . \ G = 90000 \end{cases}$

Put $r + e = a$

Then $\left.\begin{cases} r^4 + 3rrre + 6rree = a^4 \\ brrr + 3brre + 3bree = baaa \\ crr + 2cre + cee = caa \\ -dr - de = -da \end{cases}\right\} = G = 90000$

Let $r = 10$

Then

$$\text{Then}\left\{\begin{array}{l}+10000+4000e+600ee\\+40000+12000e+1200ee\\+75100+15020e+751ee\\-90000-9000e\end{array}\right\}=G=90000$$

That is, $35100+22020e+2551ee=90000$

Hence it will be $22020e+2551ee=54900$

Consequently, $8{,}63e+ee=21{,}52=D$

And $\dfrac{D}{8{,}63+e}=e$

Operation, 8,36)	21,52	
$+e=2{,}1$	20	
1. *Divisor* $=10$	1,52	First $r=10$
2. *Divisor* $=10{,}7$	1,07	$+e=2{,}1$
	45 &c.	$r+e=12{,}1=r$

for a second Operation, which being involved, and multiplied into the *Coefficients*, as before, will produce these Numbers:

$$\left.\begin{array}{l}+21435{,}8881+7086{,}24e+878{,}46ee\\+70862{,}4400+17569{,}20e+1452{,}00ee\\+109953{,}9100+18174{,}20e+751{,}00ee\\-108900{,}0000-9000{,}00e\end{array}\right\}=C.$$

Viz. $93352{,}2381+33829{,}64e+3081{,}46ee=90000$

Here, because $93352{,}2381>90000$ therefore $12{,}1>a$, and therefore it must be made $r-e=a$, which will produce the same Numbers, only all the second Signs must be changed.

Thus, $93352{,}2381-33829{,}64e=3081{,}46ee=90000$ from whence will arise this Equation:

$$+33829{,}64e-3081{,}46ee=3352{,}2381$$

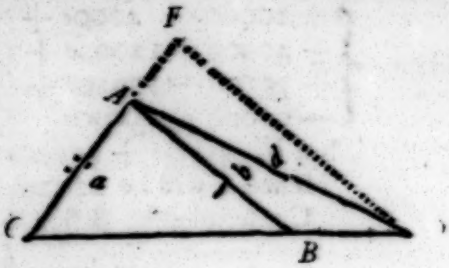
Consequently, $10{,}9784e-ee=1{,}08787332=D$

Operation 10,9784)	1,08787332	$(0{,}0999=e$
$-e=,0999$	9792	
1. *Divisor* 10,88	108673	Last $r=12{,}1$
2. *Divisor* 10,879	97911	$-e=0{,}0999$
3. *Divisor* 10,8785	1076232	$r-e=12{,}0001=a$
	979065	
	&c.	

PROBLEM XX.

In the Oblique-angled Triangle CAD, there is given the Side AD, and the Sum of the Sides AC + CD; also within the Triangle is given the Line AB perpendicular to the Side CA; thence to find the Side CA, &c.

Let



Let	1	$CA+CD=s=51$
	2	$AD=d=32$
	3	$AB=b=21$
And	4	$CA=a$ sought.
Then	5	$s-a=CD$
Suppose		the *Line* DF *parallel* to AB; CA being produced to F
Then		$\triangle\, CAB$, *and* $\triangle\, CFD$ will be alike.
And	6	$BC : CA : DC : CF$
But	7	$BC=\sqrt{bb+aa}$. Let $AF=e$, and $FD=y$
6, 7	8	$\sqrt{bb+aa} : a :: s-a : a+e$
8 ∵	9	$\dfrac{sa-aa}{\sqrt{:bb+aa}}=a+e$
5 $\bullet^2$	10	$ss-2sa+aa=\square CD$
Per *Fig.*	11	$ss-2sa+aa=aa+2ae+ee+yy=\square CF+\square FD$
11 − aa	12	$ss-2sa=2ae+ee+yy$
But	13	$dd=ee+yy=\square AF+\square FD$
12 − 13	14	$ss-2sa-dd=2ae$
Let	15	$2x=ss-dd$
14, 15	16	$x-sa=ae$
16 ÷ a	17	$\dfrac{x-sa}{a}=e$
17 + a	18	$\dfrac{x-sa+aa}{a}=a+e$
9 $\bullet^2$	19	$\dfrac{ssaa-2saaa+a^4}{bb+aa}=\square\,\overline{a+e}$
18 $\bullet^2$	20	$\dfrac{xx-2xsa+2xaa+ssaa-2sa^3+a^4}{aa}=\square\,\overline{a+e}$
19, 20	21	$\dfrac{ssaa-2saaa+a^4}{bb+aa}=\dfrac{xx-2xsa+2xaa+ssaa-2sa^3+a^4}{aa}$

This Equation being brought out of the Fractions, and into Numbers, will become $-2018a^4+125409a^2-2464230{,}25a^2+35468307a=274183922{,}25$; which being divided by 2018, the Co-efficient of the highest Power of a, will be $-a^4+62{,}1456a^3-1221{,}125a^2+17575{,}9697a=135869{,}138875$, &c.

And

And from hence the *Value* of a may be found, as in the *last Problem*, due Regard being had to the *Signs* of every *Term*.

This Work of *reducing*, or preparing *Equations* for a *Solution* by *Division*, hath always been taught both by *antient* and *modern* Writers of *Algebra*, as a Work so necessary to be done, that they do not so much as give a Hint at the *Solution* of any *adfected Equation* without it.

Now it very often happens, that, in *dividing* all the *Terms* of an *Equation*, some of their *Quotients* will not only run into a long *Series*, but also into imperfect *Fractions* (as in the *Equation* above) which renders the *Solution* both tedious and imperfect.

To remedy that *Imperfection*, I shall here shew how this *Equation* (and consequently any other) may be resolved without such *Division* or *Reduction*.

Let $b = 2018$, $c = 125409$, $d = 2464230,25$
$f = 35468307$. And $G = 274183922,25$
Then the *precedent Equation* will stand thus:

$$-baaaa + caaa - daa + fa = G$$

Put $r + e = a$ as before.

$$\text{Then will} \left\{ \begin{array}{l} -br^4 - 4brrre - 6brree = -ba^4 \\ +cr^3 + 3crre + 3cree = +ca^3 \\ -drr - 2dre - dee = -daa \\ +fr + fe \ldots\ldots = +fa \end{array} \right\} = G$$

This is plain and easily conceived. The next Thing will be, how to estimate the first *Value* of r; and, to perform that, let G be divided by b, only so far as to determine how many Places of whole *Numbers* there will be in the *Quotient*; consequently, how many *Points* there must be *(according to the Height of the Equation.)*

$$\begin{array}{l} \text{Thus } b = 2018 (\; G = 274183922,25 \; (130000 \\ \qquad\qquad\qquad\qquad 2018 \\ \qquad\qquad\qquad\qquad \overline{} \\ \qquad\qquad\qquad\qquad 7238, \ \&c. \end{array}$$

Now from hence one may as easily guess at the Value of r, as if all the *Terms* had been *divided*. That is, I suppose $r = 10$, which being involved, &c. as the *Letters above direct*, will be

$$\left.\begin{array}{l} -\ 20180000 - 8072000e - 1210800ee \\ +\ 125409000 + 37622700e + 3762270ee \\ -\ 246423025 - 49284605e - 2464230,25ee \\ +\ 354683070 + 35468307e \end{array}\right\} = G$$

Viz. $213489045 + 15734402e + 87239{,}75ee = 2741839$ &c.

Hence $15734402e + 87239{,}75ee = 60694877{,}25$

Consequently, $180{,}3e + ee = 695{,}72 = D$

And $\dfrac{D}{180{,}3 + e} = e$

Operation	180,3)	695,72	(3,7 = e
+e =	3,7)	549	
1. *Divisor*	= 183	146,72	First r = 10
2. *Divisor*	= 184,0	128,80	+ e = 3,7

&c. $r + e = 13{,}7 = r$ for a second *Operation*, with which you may proceed, as in the last *Problem*, and so on to a third *Operation*, if Occasion require such Exactness. But this may be sufficient to shew the Method of resolving any *adfected Equation*, without reducing it; which is not only very exact, but also very ready in Practice, as will fully appear in the last Chapter of this Part, concerning the *Periphery* and *Area* of the *Circle*, &c. wherein you will find a farther Improvement in the *Numerical Solution* of *High Equations* than hath hitherto been published.

CHAP. V.

Practical PROBLEMS, *and* RULES *for finding the* SUPERFICIAL CONTENTS, *or* AREAS *of Right-lin'd Figures.*

BEfore I proceed to the following *Problems*, it may be convenient to acquaint the Learner, that the *Superficies* or *Area* of any Figure, whether it be *Right-lined* or *Circular*, is composed or made up of *Squares*, either greater or less, according to the different Measures by which the Dimensions of the Figures are taken or measured.

That is, if the Dimensions are taken in Inches, the *Area* will be composed of *square Inches*; if the Dimensions are taken in Feet, the *Area* will be composed of *square Feet*; if in Yards, the *Area* will be *square Yards*; and if the Dimensions are taken by *Poles* or *Perches*, (*as in Surveying of Land*, &c.) then the *Area* will be *square Perches*, &c. These Things being understood, and the *Definitions*

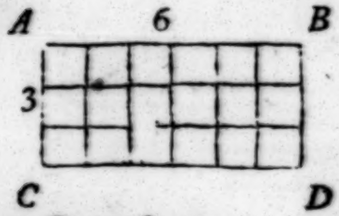
Definitions in the 283 and 284 Pages well considered, will help to render the following *Rules* very easy.

PROBLEM I.

To find the Superficial Content, or Area of a SQUARE; *or of any Right-angled* PARALLELOGRAM.

RULE { *Multiply the Length into its Breadth, and the Product will be the Area required.* (*See* Lemma 1, Page 302.)

Example, Suppose the *Line* $AB = 6$ *Yards*, and the *Breadth* AC or $BD = 3$ *Yards*, then $AB \times AC = 6 \times 3 = 18$ will be the Number of *square Yards* contained in the *Area* of the *Parallelogram* $ABCD$. This is so evident by the *Figure* only, that it needs no *Demonstration*.

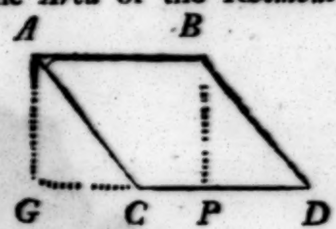


PROBLEM II.

To find the Area of any Oblique-angled Parallelogram, viz. *either of a* RHOMBUS *or* RHOMBOIDES.

RULE { *Multiply the Length into its perpendicular Height (or Breadth) and the Product will be the Area required.*

That is, the Side $AB \times BP =$ the *Area* of the *Rhombus* $ABCD$. For if BP be drawn *perpendicular* to CD, and AG be made *parallel* to BP, then will $GC = PD$ and $GP = CD$. Consequently $\triangle AGC = \triangle BPD$, and $\square ABGP =$ *Rhombus* $ABCD$. But $AB \times BP = \square ABGP$. Therefore $AB \times BP$, or $CD \times BP =$ the *Area* of the *Rhombus* $ABCD$.



Example, Suppose the Side $AB = 23$ *Inches*, and the Perpendicular $BP = 17{,}5$ *Inches*, (being the shortest or nearest Distance between the two Sides, AB, and CD.) then $AB \times BP = 23 \times 17{,}5 = 402{,}5$ *square Inches*, being the *Area* of the *Rhombus* required.

The like may be done for any *Rhomboides* whose *Length* and perpendicular *Breadth* is given.

 PRO-

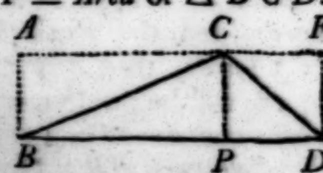
PROBLEM III.

To find the Superficial Content, or Area of any plain TRIANGLE.

Every plain Triangle is equal to half its circumscribing Parallelogram, (41. *e.* 1.) *which affords the following Rule.*

RULE { *Multiply the* Base *of the given Triangle into half its perpendicular Height, or half the* Base *into the whole Perpendicular, and the Product will be the Area.*

That is, $BD \times \frac{1}{2}CP$, or $\frac{1}{2}BD \times CP =$ *Area* of $\triangle\ BCD$.



For $AC = BP$, $AB = CP$, and BC is common to both $\triangle\ \triangle$; therefore $\triangle\ ABC = \triangle\ BCP$, and for the like Reasons, $\triangle\ CFD = \triangle\ CPD$. Therefore $\triangle\ BCP + \triangle\ CPD = \frac{1}{2}\square\ ABFD$. Consequently $\frac{1}{2}BD \times CP$, or $BD \times \frac{1}{2}CP$ will be the *Area* of $\triangle\ BCD$.

Example, Suppose the *Base* $BD = 32$ *Inches*, and the perpendicular *Height* $CP = 14$ *Inches*.

Then $\frac{1}{2}BD \times CP = 16 \times 14 = 224$. Or $BD \times \frac{1}{2}CP = 32 \times 7 = 224$. Or thus, $32 \times 14 = 448$. Then 2) 448 (224 = the *Area* of the *Triangle* BCD in *square Inches*.

PROBLEM IV.

To find the Superficies, or Area of any TRAPEZIUM.

First, divide the given *Trapezium* into *two Triangles*, by drawing a Diagonal from one of its acute *Angles* to the opposite *Angle*; and let fall two *Perpendiculars* (from the other two Angles) upon the *Diagonal*, as in the following *Figure*. Then

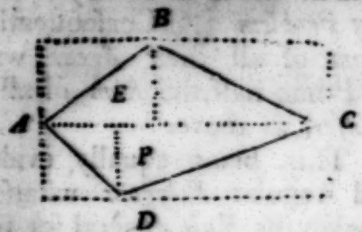
RULE { *Multiply half the Diagonal into the Sum of the two Perpendiculars, or half the Sum of the Perpendiculars into the Diagonal, and the Product will be the Area.*

That is, $\frac{1}{2}\ \overline{AC \times BP + ED}$. Or $AC \times \overline{\frac{1}{2}BP + \frac{1}{2}ED} =$ *Area* of the *Trapezium* $ABCD$.

For the $\triangle\ ABC$ is half its circumscribing Parallelogram; and the $\triangle\ ACD$ is also half of its circumscribing Parallelogram, as hath been proved at the last *Problem*.

Consequently,

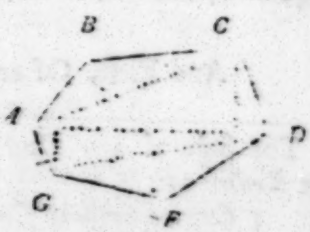
Consequently, $\overline{BP+ED}\times\frac{1}{2}AC$, or $\frac{1}{2}BP\overline{+\frac{1}{2}ED}\times AC$ will be the *Area* of the *Trapezium*, as above.



Example. Suppose the *Diagonal* $AC=33$ *Feet*, and the *Perpendicular* $BP=15$ *Feet*, and the *Perpendicular* $ED=14$ *Feet*. Then $BP+ED=29$ *Feet*, and $\overline{BP+ED}\times\frac{1}{2}AC=29\times16,5$ $=478,5$. Or $AC\times\overline{\frac{1}{2}BP+\frac{1}{2}ED}=33\times{}^{29}=478,5$. Or thus, $29\times33=957$. Then 2) 957 (478,5 any of these *Products* are the *Area* of the *Trapezium ABCD*.

PROBLEM V.

To find the Superficial Content or Area of any irregular Polygon or *many sided Figure, which by some Authors is called a Triangulate, because (as I suppose) it must be divided into Triangles,* as in the annexed *Figure ABCDFG*; by which it is evident, that the Sum of the *Areas* of all those *Triangles*, found as in the last *Problem*, &c. will be the *Area* of their circumscribing *Polygon*.



PROBLEM VI.

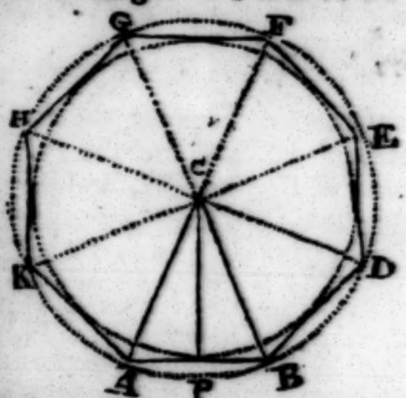
To find the Superficies, or Area of any regular Polygon, viz. *of any regular* PENTAGON, HEXAGON, HEPTAGON, OCTAGON, &c.

General RULE. { *Multiply half the Sum of its Sides into the Radius of the inscribed Circle, or half the said Radius into the Sum of the Sides, and the Product will be the Area required.*

That is, $\frac{AB+BD+DE+EF+FG+GH+HK+KA}{2}:\times CP$ = the *Area* of the annexed *Octagon*; wherein it is evident, that its *Area* is composed of so many equal *Isosceles Triangles* as there are Numbers of Sides in the *Polygon*, viz. of eight *Isosceles Triangles*, whose *Bases* are the Sides of the *Octagon*, viz. $AB=BD$ $=DE$, &c. And the Sides of those *Triangles*, CA, CB, CD, &c. are the *Radius*'s of the circumscribing *Circle*; and their perpendicular Heights, viz. CP, is the *Radius* of the inscribed Circle.

But

But the *Area* of any one of those *Triangles* is $\frac{1}{2} AB \times CP$ by *Problem* 3. Consequently the *Sum* of all their *Areas* will be CP into half the *Sum* of all their *Bases*, as above.



This, being equally evident in all regular *Polygons* whatsoever, makes the *Rule* general for finding their *Areas*.

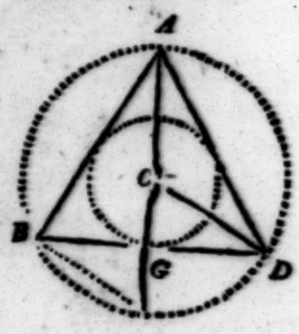
Now, because it is required to have the *Radius* of the proposed *Polygon's* inscribed *Circle*. I shall here insert (and demonstrate) the *Proportions* that are between the Sides of several regular *Polygons* and the *Radius*'s both of their inscribed and circumscribing *Circles*; the one will help to delineate or project the *Polygon* (if Occasion require it) and the other will help to find its *Area*.

And First, Of an Equilateral Triangle.

The Side of any *Equilateral plain Triangle* is in *Proportion* to the *Radius* of

$$\text{its}\begin{cases}\textit{Circumscribing Circle,}\\ \textit{Inscribed Circle,}\\ \textit{Perpendicular Height,}\end{cases}\text{As } 1 : \text{To}\begin{cases}0{,}57735027 \text{ \&c.}\\ 0{,}28867513 \text{ \&c.}\\ 0{,}86602540 \text{ \&c.}\end{cases}$$

$$\text{i. e.}\begin{cases}AB : CD :: 1 : 0{,}57735027\\ AB : CG :: 1 : 0{,}28867513\\ AB : AG :: 1 : 0{,}86602540\end{cases}$$



Demonstration.

Let $AB = BD = 1$, then will $BG = GD = 0{,}5$; but $\square\, AB - \square\, BG = \square\, AG$ by *Theorem* 11. That is, $1 - 0{,}25 = 0{,}75 = \square\, AG$, consequently, $\surd\, 0{,}75 = 086602540 = AG$: Then $AG : BA :: B : AH$, by *Theorem* 13, that is, $0{,}8660254 : 1 :: 1 : 1{,}5470054$, &c. $= AH$, then $\frac{1}{2} AH = 0{,}57735027 = CA$. Again, $AG : DG :: DG : CG$, that is, $0{,}8660254 : 0{,}5 :: 0{,}5 : 0{,}28867513 = CG$. Q. E. D.

Now, by the Help of the First of these *Proportions*, it will be easy to resolve the following Problem.

PRO-

PROBLEM VII.

The Side of any Equilateral plain Triangle being given, to find its Area.

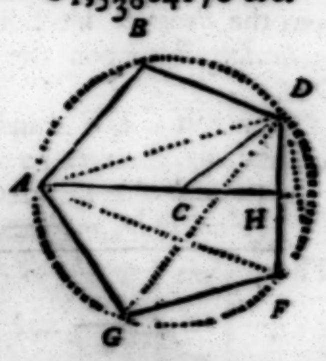
Example, Suppose the Side of the proposed *Triangle* ABC to be 25 *Inches*, viz. $AB = BC = CA = 25$. First $1 : 0{,}8660254 :: AB = 25 : 21{,}650635 = BP$ by *Theorem* 13. Then $AP (= \frac{1}{2} CA) \times BP =$ the *Area* of $\triangle ABC$ by *Rule* to *Problem* 3, that is, $12{,}5 \times 21{,}650635 = 270{,}6329$ the *Area* in square *Inches*.

Or this Problem may be otherwise *resolv'd*, thus: Let $b = AP = \frac{1}{2} AC$. Then $2b = AB$. But $\square AB - \square AP = \square BP$. By *Theorem* 11. That is, $4bb - bb = 3bb = \square BP$. Consequently, $\sqrt{3bb} = BP$. Then $b\sqrt{3bb} = BP \times \frac{1}{2} AC$. viz. $\sqrt{3bbbb} =$ the *Area* of the *Triangle*.

Secondly, For a PENTAGON.

The Side of any Pentagon is in Proportion to the Radius of its { *Circumscribing Circle*, *Inscribed Circle*, *Perpendicular Height*, } As 1 : To { 0,85065080 &c. 0,68819096 &c. 1,53884176 &c. }

Viz. { $AB : AC :: 1 : 0{,}85065080$; $AB : CH :: 1 : 0{,}68819096$; $AB : AH :: 1 : 1{,}53884176$ }



DEMONSTRATION.

Let $AB = 1$. And draw the Diagonals AD, AF, and DG, which will be equal to one another. Then will $AG \times DF + AD \times GF = AF \times DG$ by *Theorem* 19. Consequently, $AG \times DF = AF \times DG : - AD \times GF$, that is, $\square AB = \square AD : - AD \times GF = 1$ (because $AB = AG = DF$, and $AD = AF = DG$) hence it will be $AD = 1{,}61803398$, then $\square AD - \square DH = \square AH$ by *Theorem* 11. But $DH = \frac{1}{2} AB$, therefore $\sqrt{\square AD - \frac{1}{4} \square AB} = AH = 1{,}53884176$. Again, $AH : AD :: AD : AX = 2AC$. For $\triangle AHD$ and $\triangle ADX$ are alike.

Ergo

Ergo $\frac{\square AD}{AH} = 2AC = 1,70130161$. Hence $AC = 0,85065080$
But $AH - AC = CH = 0,68819096$, &c. Q. E. D.
From hence it will be easy to resolve the following *Problem*.

PROBLEM VIII.

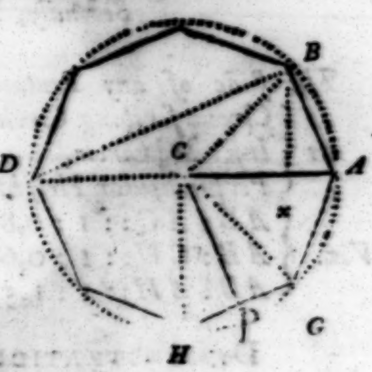
The Side of any regular Pentagon being given, to find its Area.

Example, Suppose the given Side to be 15 *Inches* long, then it will be, as $1 : 1,538841 76 :: 15 : 22,0826264$ the *perpendicular Height*; and by the general Rule $22,0826264 \times \frac{15}{2} = 165,619698$ the *Area* required.

Thirdly, For an OCTAGON.

The Side of any regular Octagon is in Proportion to the Radius of its $\begin{cases} \textit{Circumscribing Circle}, \text{ As } 1 : \text{to } 1,30656296 \text{ \&c.} \\ \textit{Inscribed Circle}, \text{ As } 1 : \text{to } 1,20710678 \text{ \&c.} \end{cases}$

Viz. $\begin{cases} BA : CA :: 1 : 1,30656296 \\ BA : CP :: 1 : 1,20710678 \end{cases}$



DEMONSTRATION.

Draw the *Right-line* DB, and from the *Point* B let fall the *Perpendicular* Bx upon the *Diameter* DA.

Then will $\triangle DBA$ and $\triangle DxB$ be alike, by *Theorem* 10 and 12.

Let $\begin{cases} b = BA = 1 . a = CA \\ e = BD, \text{ and } y = Bx \end{cases}$

Then	1	$2a : b :: e : y$. viz. $DA : BA :: DB : Bx$
$1 \because$	2	$\frac{2ay}{b} = e = DB$
$2 \odot^2$	3	$\frac{4aayy}{bb} = ee = \square DB$
But	4	$4aa - \frac{4aayy}{bb} = bb$
That is	5	$\square DA - \square DB = \square BA$. By *Theorem* 11.
$4 \times bb$		$4bbaa - 4aayy = bbbb$.
Again	6	$\begin{cases} \frac{1}{2}aa = yy. \text{ For } Cx = B \,\because \\ \text{and } \square Cx + \square Bx = \square CB = aa \end{cases}$

5, 6	7	$4bbaa - 2a^4 = b^4$. Or $2a^4 - 4bbaa = -b^4$
$7 \div 2$	8	$aaaa - 2bbaa = -\frac{1}{2}b^4$
8 C □	9	$a^4 - 2bbaa + b^4 = b^4 - \frac{1}{2}b^4 = \frac{1}{2}b^4$
9 ww^2	10	$aa - bb = \sqrt{\frac{1}{2}b^4}$
10 + bb	11	$aa = bb + \sqrt{\frac{1}{2}b^4}$
11 ww^2	12	$a = \sqrt{: bb + \sqrt{\frac{1}{2}b^4}} = 1,30656296$, &c. $= CA$
Then	13	$aa - \frac{1}{4}bb = \square CP$, viz. $\square CH - \square HP = \square CP$
13 ww^2	15	$\sqrt{aa - \frac{1}{4}bb} = 1,20710678$ &c. $= CP$.

From hence it will be easy to find the *Area* of any *Octagon.*

PROBLEM IX.

The Side of any regular Octagon *being given, to find its* Area.

Example, Suppose the Side given to be 12 Inches long; *First,* as $1 : 1,20710678 :: 12 : 14,48528136 =$ the *Radius* of its inscribed Circle; then $12 \times 4 = 48$ is half the Sum of its Sides, and $48 \times 14,48528136 = 695,2935$ the *Area* required.

Fourthly, For a Decagon.

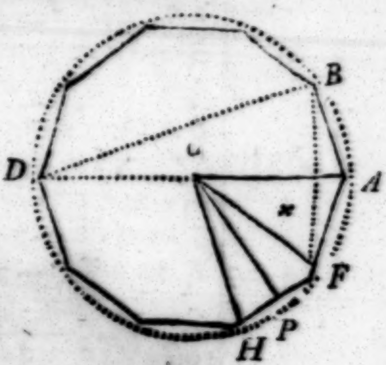
The Side of any regular Decagon *(viz. a* Polygon *of ten equal Sides) is in Proportion to the* Radius *of*

Its { *Circumscribing Circle,* as 1 : to 1,61803398 &c.
{ *Inscribed Circle,* as 1 : to 1,53884176 &c.

Viz. { $BA : CA :: 1 : 1,61803398$
{ $BA : CP :: 1 : 1,53884176$

Demonstration.

Let { $b = BA = 1$. $a = CA$
{ $e = DB$, and $y = Bx$



Then That is,	1	$2a : b :: e : y$ $DA : BA :: DB : Bx$
1 ∵	2	{ $2ay = be$ and $2y = \frac{be}{a}$
But	3	$2y : e :: 1 : 1,61803398$. See *Pentagon.*
3 ∵	4	$\frac{1e}{1,61803398} = 2y = \frac{be}{a} = \frac{1e}{a}$
$4 \div 1e$	5	$1,61803398 = a = CA$
Again	6	{ $aa - \frac{1}{4}bb = \square CP$. viz. $\square CF - \square PF = CP$. By *Theorem* 11.
That is,	7	$\sqrt{2,61803398 - 0,25} = 1,53884176 = CP$.

PROBLEM X.

The Side of any regular Decagon *being given, to find its* Area.

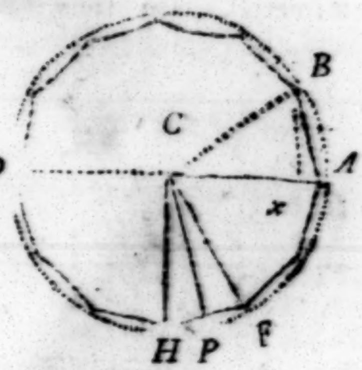
Example, Let the given Side be 14 Inches long; then, as 1 : 1,538841 76 :: 14 : 21,543784 = the *Radius* of the inscribed Circle; and 14 × 5 = 70 is half the Sum of its Sides. Lastly, 21,543784 × 70 = 1508,06488 the *Area* required.

Fifthly, For a Dodecagon.

The Side of any regular Dodecagon (viz. a Polygon *of twelve equal Sides) is in Proportion to the* Radius *of*

its { *Circumscribing Circle*, as 1 : to 1,93185165, &c.
Inscribed Circle, as 1 : to 1,86632012, &c.

Viz. { $BA : CA :: 1 : 1,93185165$
$BA : CP :: 1 : 1,86632012$



Demonstration.

Let $b = BA = 1$. $a = CA$ as before
And $e = xA$; then $a - e = Cx$

First	1	$\{ bb - \Box Bx = ee$ By Figure.
But	2	$Bx = \frac{1}{2} CA = \frac{1}{2} a$
2 $\odot^2$	3	$\Box Bx = \frac{1}{4} aa$
1, 3	4	$bb - \frac{1}{4} aa = ee$
4 ww²	5	$\sqrt{bb - \frac{1}{4} aa} = e$
Again	6	$aa - \frac{1}{4} aa = aa - 2ae + ee$
Viz.		$\Box CB - \Box Bx = \Box Cx$
5 × 2a	7	$2a \sqrt{bb - \frac{1}{4} aa} = 2ae$
4 — 7	8	$bb - \frac{1}{4} aa - 2a\sqrt{bb - \frac{1}{4} aa} = ee - 2ae$
7, 8	9	$aa - \frac{1}{4} aa = aa + bb - \frac{1}{4} aa - 2a\sqrt{bb - \frac{1}{4} aa}$
9 +	10	$2a\sqrt{bb - \frac{1}{4} aa} = bb$
10 $\odot^2$	11	$4bbaa - aaaa = b^4$
11 +	12	$aaaa - 4bbaa = -b^4$
13, C □	13	$aaaa - 4bbaa + 4b^4 = 3b^4 = 3$
1 ww²	14	$aa - 2bb = \sqrt{3} = 1,7320508075$
14 + 2bb	15	$aa = 2bb + \sqrt{3} = 3,7320508075$
15 ww²	16	$a = \sqrt{3,7320508075} = 1,93185165 = CA$
Again	17	$aa - \frac{1}{4} bb = \Box CP$. viz. $\Box CF - \Box PF = \Box CP$
17, Hence	18	$CP = \sqrt{aa - \frac{1}{4} bb} = 1,86632012$. Q.E.D.

Con-

Consectary.

Hence if the Side of any regular *Dodecagon* be given, the *Radius* of its inscribed Circle may be easily obtained, and thence the *Area* found; as in the last Problem.

The Work of the 'foregoing *Polygons*, being well considered, will help the young *Geometer* to raise the like Proportions for others, if his Curiosity requires them: And not only so, but they will also help to form a true Idea of a Circle's *Periphery* and *Area*, according to the Method which I shall lay down in the next Chapter for finding them both.

CHAP. VI.

A new and easy Method of finding the CIRCLE'S PERIPHERY *and* AREA *to any assigned Exactness (or Number of Figures) by one Equation only. Also a new and facile Way of making Natural* SINES *and* TANGENTS.

LET us suppose (*what is very easy to conceive*) the Circle's *Area* to be composed or made up of a vast Number of plain *Isosceles Triangles*, having their acutest *Angles* all meeting in the Circle's Center. And let us imagine the *Bases* of those *Triangles* so very small, that their Sides and their Perpendicular Heights, *viz.* the *Radius*'s of their circumscribed and inscribed Circles (*vide* Problem 6.) may become so very near in *Length* to each other, as that they may be taken one for another without any sensible Error: Then will the *Peripheries* of their circumscribing and inscribed Circles become (altho' not co-incident, yet) so very near to each other, as that either of them may be indifferently taken for one and the same Circle.

But how to find out the Sides of a *Polygon* (viz. the Bases of those Isosceles Triangles) to such a convenient Smallness as may be necessary to determine and settle the Proportion betwixt a Circle's *Diameter* and its *Periphery* (to any assigned Exactness) hath hitherto been a Work which required great Care and much Time in its Performance; as may be easily conceived from the Nature of the Method used by all those who have made any considerable Progress in it, viz. *Archimedes*, *Snelius*, *Hugenius*, *Mœtius*, *Van Culen*, &c. These proceeded with the bisecting of an *Arch*, and found the Value of its Chord to a convenient Number of Figures

at every ſingle Biſection, repeating their Operations until they had approached to the *Chord* deſigned.

And this Method is made Choice of by the learned Dr. *Wallis* in his Treatiſe of *Algebra*; wherein, after he hath given us a large Account of the different Enquiries made by ſeveral (very eminent in Mathematical Sciences) in order to find out ſome eaſier and more expeditious Way of approaching to the Circle's *Periphery*, as in Chap. 82, 84, 85, 86, and ſeveral other Places, he comes to this Reſult, (Page 321.)

“ 'Tis true, *ſaith he*, we might in like Manner proceed by “ continual Triſection, Quinquiſection, or other Section, if we “ had for theſe as convenient Methods of Operation as we have “ for Biſection: But becauſe *Euclid* ſhews how to biſect an Arch “ Geometrically, but not to triſect, *&c.* and the one may be “ done *(Algebraically)* by reſolving a Quadratic Equation, but “ not thoſe other, without Equations of a higher Compoſition, “ I therefore make Choice of a continual Biſection, *&c.*”

And then he lays down theſe following Canons.

The Subtenſe of $\frac{1}{6}$	1	into	6
of $\frac{1}{2}$	$\surd:2-\surd 3$	into	12
of $\frac{1}{24}$	$\surd:2+\surd:2+\surd 3$	&c.	24
of $\frac{1}{48}$	$\surd:2+\surd:2+\surd:2+\surd 3$		48
of $\frac{1}{96}$	$\surd:2-\surd:2+\surd:2+\surd:2+\surd 3$		96
&c.	$\surd:2-\surd:2+\surd:2+\surd:2+\surd:2+\surd 3$		192
	$\surd:2-\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd 3$		384
	$\surd:2-\surd:2-\surd:2+\surd:2+\surd:2+\surd:2+\surd:2+\surd 3$		768
	&c.		&c.

“ How tedious and troubleſome the Work of theſe complicat- “ ed *Extractions* is, I leave to the Conſideration of thoſe, who “ either have had Experience therein, or out of Curioſity will “ give themſelves the trouble of making Trial.

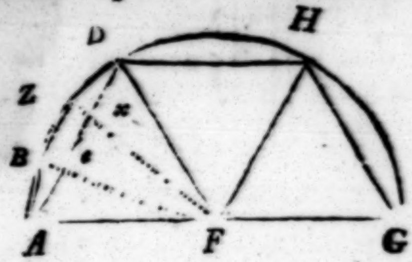
Again, in Page 347, the Doctor inſerts a particular Method propoſed by *Libnitius*, publiſhed in the *Acta Eruditorum* at *Leipſic*, for the Month of *February* 1682, in order to find the Circle's *Area*, and conſequently its *Periphery*, which is this:

As 1 : to $\frac{1}{2}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}$, &c. infinitely : : ſo is the Square of the *Diameter* to the Circles's *Area*. But this convergeth ſo very ſlowly, that it is not worth the Time to purſue it.

I ſhall here propoſe a new Method of my own, whereby the Circle's *Periphery*, and conſequently its *Area*, may be obtained infinitely near the Truth, with much greater Eaſe and Expedition

tion than either that of *Bisection*, or that of *Libnitius*, as above, or any other Method that I have yet seen; it being performed by resolving only one Equation, deduced by an easy Process from the Property of a Circle, (known to a Cooper) which is this:

The Radius *of every Circle is equal to the* Chord *of one sixth Part of its* Periphery. That is, $AD = DH = HG$, the Chords of one third Part of the *Semicircle*, are each equal to AF its *Radius*. Then if the Arch AD be trisected, it will be $AD = BZ = ZD$.



Let $\begin{cases} R = AF = 1 \\ c = AD = 1 \\ a = AB. \text{ Quære } a. \end{cases}$

Then	1	$R : a :: a : \frac{aa}{R} = Be$
And	2	$R : a :: R - \frac{aa}{R} : c - 2a$
That is, For	3	$FB : BZ :: Fe : ex = AD - 2a$ $\triangle AFB$, and $\triangle BAe$, are alike And $AB = Ae = Dx$, &c.
2 ∵	4	$Rc - 2Ra = Ra - \frac{aaa}{R}$
4 × &c.	5	$3R^2a - aaa = RRc$. That is, $3a - aaa = 1$ Here $a =$ the Chord of $\frac{1}{18}$ Part of the *Circle*. For $\frac{1}{3}$ of $\frac{1}{6} = \frac{1}{18}$.

Next, *To trisect the* Arch AB.

Let	1	$3y - y^3 = a$ the last Chord.
1 ⊙ 3	2	$27y^3 - 27y^5 + 9y^7 - y^9 = a^3$
1 × 3	3	$9y - 3y^3 = 3a$
3 − 2	4	$9y - 30y^3 + 27y^5 - 9y^7 + y^9 = 3a - a^3 = 1$ Here $y =$ the Chord of $\frac{1}{54}$ Part of the Circle.

Again, *To trisect the* Arch *whereof* y *is the* Chord.

Let	1	$3a - a^3 = y$
1 ⊙ 3	2	$27a^3 - 27a^5 + 9a^7 - a^9 = y^3$
1 ⊙ 3	3	$243a^5 - 405a^7 + 270a^9 - 90a^{11} + 15a^{13} - a^{15} = y^5$

1 ⊙ 7

$1 \odot^7$	4	$\left\{\begin{array}{l}2187a^7 - 5103a^9 + 5103a^{11} - 2835a^{13} + \\ 945a^{15} = y^7\end{array}\right.$
$1 \odot^9$	5	$\left\{\begin{array}{l}19683a^9 - 59049a^{11} + 78732a^{13} - \\ 61236a^{15} = y^9\end{array}\right.$
$1 \times \overline{9}$	6	$27a - 9a^3 = 9y$
$2 \times \overline{30}$	7	$810a^3 - 810a^5 + 270a^7 - 30a^9 = 30y^3$
3×27	8	$\left\{\begin{array}{l}6561a^5 - 10935a^7 + 7290a^9 - 2430a^{11} + \\ 405a^{13} + 27a^{15} = 27y^5\end{array}\right.$
$4 \times \overline{9}$	9	$\left\{\begin{array}{l}19683a^7 - 45927a^9 + 45927a^{11} - \\ 25515a^{13} + 8505a^{15} = 6y^7\end{array}\right.$
$\left.\begin{array}{r}6 - 7 \\ +8 - 9 \\ +5\end{array}\right\}$	10	$\left\{\begin{array}{l}27a - 819a^3 + 7371a^5 - 30888a^7 + \\ 72930a^9 - 107506a^{11} + \\ 104652a^{13} - 69768a^{15}\end{array}\right\} = 1$

Here $a =$ the *Chord* of $\frac{1}{162}$ Part of the Circle.

Proceeding on in this Method of continually trisecting the Arch of every new Chord and still connecting the produced *Equations* into one, as in the two last Trisections, 'twill not be difficult to obtain the *Chord* of any assigned Arch, how small soever it be.

Now, in order to facilitate the Work of raising these *Equations* to any considerable Height, 'twill be convenient to add a few useful Observations concerning their Nature, and of such Contractions as may be safely made in them; which, being well understood, will render the Work very easy.

" 1. I have observed, that every *Trisection* will gain or advance one Figure in the Circle's Periphery, but no more. Therefore so many Places of Figures as are at first designed to be perfect in the Periphery, so many Trisections must be repeated to raise an Equation that will produce a Chord answerable to that Design.

" 2. I have also found, that all the superior Powers (of a) whose Indices are greater than the Number of Trisections, (viz. whose Indices are greater than the Number of designed Figures) may be wholly rejected as insignificant.

" 3. When once the Number of Trisections and thence the highest Power (of a) is determined, the third Process (viz. the third Trisection) may be made a fixed or constant Canon; for by it, and Multiplication only, all the succeeding Trisections (how many soever they are) may be compleated without repeating the several Involutions.

" 4. In

"4. In raiſing and collecting the Co-efficients of the ſeveral Powers (of *a*) it will be ſufficient to retain only ſo many ſignificant Figures (at a^3) as there is deſigned to be Places of Figures in the *Periphery* (or at moſt but two more) and every ſucceeding ſuperior Power may be allowed to decreaſe two Places of ſignificant Figures: But herein great Care muſt be taken to ſupply the Places, of thoſe Figures that are omitted, with Cyphers, that ſo the whole and exact Number of Places may be truly adjuſted; otherwiſe all the Work will be erroneous.

"Now the Number of thoſe ſupplying Cyphers may be very conveniently denoted by Figures placed within a *Parentheſis*, thus: $576\,(8)\,a^3$, may ſignify $57600000000a^3$, as in the following Equations. The like may be done with *Decimal Parts*, thus: (,7)658 may ſignify ,0000000658 &c. which will be found very uſeful in the Solution of theſe and the like Equations."

The aforeſaid Contractions may be ſafely made, becauſe both the ſuperior Powers of *a*, which are rejected; as alſo thoſe Numbers that are omitted in the Co-efficients (and ſupplied with Cyphers) would produce Figures ſo very remote from Unity, as that they would not affect the *Chord* deſigned; that is, they would not affect the *Chord* in that Place wherein the deſigned *Periphery* is concerned; as will in Part appear in the following Example.

If theſe Directions be carefully minded, 'twill be eaſy to raiſe an *Equation* that will produce the Side of a *regular Polygon*, whoſe Number of Sides ſhall be vaſtly numerous, conſequently infinitely ſmall: But, I preſume, 'twill be ſufficient for an *Example*, to find the Side of a *Polygon* conſiſting of 258280326 equal Sides; that is, if I find the *Chord* of $\frac{1}{258280326}$ Part of the *Circle's* Periphery, and that requires but ſixteen *Triſections*, which being ordered, as before directed, will produce this *Equation*.

$$\left\{\begin{array}{l} 43046721a - 33236017948696812(4)a^3 \\ +769837653199714(20)a^5 - 849121853284{1}(35a^7 \\ +54633331143(50)a^9 - 230083348(66)a^{11} \\ +6830988(79)a^{13} - 15072(94)a^{15} \end{array}\right\} = 1$$

Here the Value of *a* will have 23 Places of Figures true; that is, the Sides of the inſcribed and circumſcribed Polygons will be exactly the ſame to 23 Places of Decimal Parts, but no farther; all which may be eaſily obtained at two Operations. And for the firſt 'twill be ſufficient to take only three Terms of the *Equation*, which will admit of being yet farther contracted, thus:

Let

$$\text{Let}\left\{\begin{array}{l}43046721\,a - 3323601794(12)a^3 \\ +76983765(27)a^5\end{array}\right\} = 1$$

And let $r+e=a$; then rejecting all the Powers of e, that arise by Involution above eee,

it will be $r^3+3rre+3ree+eee=aaa$

And $r^5+5r^4e+10r^3ee+10rreee=a^5$

Then the first single Value of r may be thus found:

43046721) 1,00000000 (,00000002 $= r$

This ,00000002 $= r$ being duly involved, and its Powers multiplied into their respective *Co-efficients*, will produce

$$\left.\begin{array}{l}+,86093442+43046721e \\ -,02658881-\ 3988322e-199416(9)ee-3324(18)eee \\ +,00024635+\quad 61587e+\quad 6159(9)ee+\ 308(18)eee\end{array}\right\}=1$$

viz. ,83459196+39119986e— 193257(9)ee—3016(18) eee=1

Hence 39119986e — 193257(2)ee—3016 (18)ee=0,16540804

All the *Terms* of this last *Equation* being divided by 193257 (9) the *Co-efficient* of ee, it will then become

,0000002024e—ee—,156(5) ee=,000000000000000 8558968=D

$$\textit{Consequently,}\quad \left\{\frac{D+156\,(5)\,eee}{,0000002024-e}\right. = e$$

Operation.

,0000002024) ,0000000000000008558968 (‘000000004=e

—e ,0000000043 ,+ 00000000000000000009984 = 156 (5) eee

1 *Di.* ,000000198) ,0000000000000008568952 (,000000004327

2 *Di.* ,0000001981

792

6489

5943

5465

3962

&c.

First r = ,00000002

+ e = ,000000004327

$r+e$ = ,000000024327 = a.

Or rather new r for a second Operation.

Now, if this first Value of a = ,000000024327 *were not continued to more Places of Figures by a second Operation, but only multiplied into the Number of* Chords, *viz.* ,000000024327×258280326 =6,28318539, &c. *the Periphery of that* Circle *whose Diameter is* 2, *nearer than either* Archimedes, *or* Mœtius's *Proportion: For* Ar-

Archimedes *makes* it 6,285714 &c. *viz. As* 7 *to* 22. *And* Mœtius *makes it* 6,28318584 &c. *viz. As* 113 *to* 355.

But if the whole *Equation* before propoſed be now taken, and we proceed to a ſecond *Operation*, the Value of a may be encreaſed with twelve Places of *Figures* more, and thoſe may be obtained by plain *Diviſion* only.

Thus, let $r + e = a$, as before, and let all the Powers of e be now rejected as inſignificant;

$$\text{Then will} \left\{ \begin{array}{l} r + e = a \\ r^3 + 3r^2e = a^3 \\ r^5 + 5r^4e = a^5 \\ r^7 + 7r^6e = a^7 \end{array} \right\} \text{ and } \left\{ \begin{array}{l} r^9 + 9r^8e = a^9 \\ r^{11} + 11r^{10}e = a^{11} \\ r^{13} + 13r^{12}e = a^{13} \\ r^{15} + 15r^{14}e = a^{15} \end{array} \right.$$

The ſeveral Powers of $r = ,000000024327$ being raiſed, and multiplied into their reſpective *Co-efficients*, will produce theſe following Numbers.

$$\left. \begin{array}{lrlr} + & 1,04719758 1767 & + & 43046721e \\ - & ,0478491965983948 65 & - & 5900751e \\ + & ,0006559064845953 55 & + & 134810e \\ - & ,0000042814404133 75 & - & 1232e \\ + & ,0000000163025178 63 & + & 6e \\ - & ,0000000000406311 67 & - & 0e \\ + & ,0000000000000713 88 & + & 0e \\ - & ,0000000000000000 93 & - & 0e \end{array} \right\} = 1$$

Viz. $1,0000000264747451 06 + 37279554e = 1$

Hence $37279554e = -,0000000264747451 06 = D$: Or rather

$-37279554e = \;,0000000264747451 06 = D$

Conſequently, $\left\{ \dfrac{D}{37279554} = -e \right.$

Operation.

$37279554)\ ,000000026474745106\ (\ \cancel{(15)7101}67967 = -e$

$$\begin{array}{r} 260956878 \\ \hline 37905730 \\ 37279554 \\ \hline 62617660 \\ 37279554 \\ \hline \end{array}$$

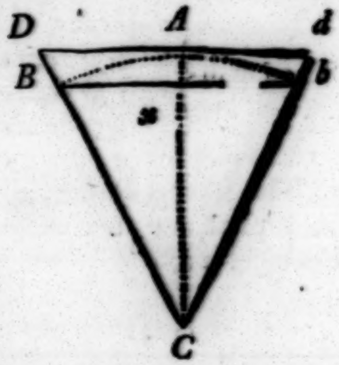
&c.

Z z Laſt

Last $r = ,000000024327$
$-e = ,0000000000000000710167967$

$r - e = ,00000002432699928983203 3 = a$ the *Chord* or *Side* of the *Polygon* required.

Then the next Work will be to examine how many Places of these *Figures* will hold true to the *Circle's Periphery:* In order to that let a be represented by the *Chord* B b, in the *annexed Scheme*; and let $Bx = xb$. Then will $Bx = \frac{1}{2}a = (,7)\,121634996449160165$. and $\Box\, BC - \Box\, Bx = \Box\, Cx$. Let the *Radius* $BC = 1$ as before. Then will the $\surd \Box BC - \Box Bx = Cx = ,9999999999999999$, &c.



But $Cx : xB :: CA : AD$ } per *Fig.*
or $Cx : Bb :: CA : Dd$ }

Ergo $Dd = (,7)\,243269992898320354$ the Side of the *Circumscribing Polygon.* Then will $a \times 258280326$ be the *Perimeter* of the *Inscribed Polygon.* And $Dd \times 258280326$ will be the *Perimeter* of the *Circumscribing Polygon.* That is, $6,28318530717958 59$ = the *Perimeter* of the *Inscribed Polygon.* And, $6,28318530717958 65$ = the *Perimeter* of the *Circumscribed Polygon.*

Hence 'tis evident, that the *Circle's Periphery*, whose *Diameter* is 2, may be concluded 6,283185307179586 4 true, because the *Perimeters* of the inscribed and circumscribed *Polygons* are so far very near being *Co-incident*, or the same.

'Tis possible there may be some who will think this is tedious and troublesome Work; but if those please to consider, that, if this *Periphery* were to be found by the aforesaid *Method* of *Bisection*, it would require these following *Extractions*.

Viz.
$\surd : 2 - \surd : 2 + \surd : 2 + \surd : 2 + \surd : 2 \surd : 2 + \surd : 2$
$+ \surd : 2 + \surd : 2 + \surd : 2 + \surd : 2 + \surd : 2 \surd : 2 + \surd : 2$
$+ \surd : 2 + \surd : 2 + \surd : 2 + \surd : 2 + \surd : 2 \surd . 2 + \surd : 2$
$+ \surd : 2 + \surd : 2 + \surd : 2 + \surd : 2 + \surd : 2 \surd : 3$ *multiplied* into 402809984.

Here the first Root (*viz.* $\surd$ 3 must be extracted at least to one hundred and two Places of Figures. The second Root (*viz.* $\surd : 2 + \surd\, 3$) must have 99 Places of Figures in it. The third

third Root (*viz.* $\sqrt{:2+\sqrt{:2+\sqrt{3}}}$) must have 96 Places in it, *&c.* every *Extraction* being allowed to decrease three Places, that so the last Root (viz. the *Chord* sought) may consist of 24 Places of Figures, as above.

I say, whoever duly considers the Trouble of these so often repeated *Extractions* will, I presume, be pleased with what I have done. For truly, when I consider the great Time and Care required in them, I cannot but admire at the Patience of the laborious *Van Culen*, who proceeded that way until he had found the *Circle's Periphery* to Thirty-six Places of *Figures*, to wit, 6,28318530717958647692528676655900576. *These Numbers are said to be engraven upon his Tomb Stone in St.* Peter's *Church in* Leyden, *for a Memorial of so great a Work.*

Having thus obtained the Circle's *Periphery*, its Arch may easily be found (to the same Number of Figures) by *Problem* 6. That is, if half the *Periphery* of any Circle be multiplied into half its *Diameter*, the Product will be that *Circle's Area*, as will appear farther on. Therefore 3,141592653589793 will be the *Area* of the *Circle* whose Diameter is 2.

Thus I have shewed the young *Geometer* how to find the *Circle's Periphery* and *Area* to what Exactness he pleases to approach: for precisely true they cannot be found, notwithstanding the late *Pretensions* of a certain *Frenchman* who hath published to the World (in the *Works of the Learned)* that after twenty-five Years Study he had found the Quadrature of the Circle: But if he had perused the 83d Chapter of Dr. *Wallis's Algebra*, he might there have seen his Error, viz. the Impossibility of what he pretended to; for it is as impossible to square the Circle (that is, *to find its true Area)* as it is to find the Root of a *Surd Number*.

Note, What I have here proposed and done by the Trisection of an Arch, may as easily and much more speedily be performed by Quinquesection or Septisection, &c. But because the Scheme for Trisection is more simple, and may be easier understood by a Learner than those of the other Sections (of which see my *Compendium of Algebra*, Pages 76 and 79) I have for that Reason made Choice of Trisection.

As to the Proportion of one Circle to another, and of the Circle to the *Ellipsis*, &c. those shall be fully shewed when we come to the fifth Part.

Before I conclude this Part I ſhall make ſome Uſe or Application of the above-found Periphery, in finding the *Quantity* of *Angles*, which is done by the Help of *Right-lines*, called *Sines* and *Tangents*, the Length whereof are calculated to every Degree and Minute of a Quadrant, by much Labour. But I ſhall here ſhew how to find the natural Sine (and conſequently the natural Tangent) of any propoſed Arch or Angle, by two Equations, without the Help of any precedent Sine, as uſual; which I did ſome Years ago communicate to the ingenious Mr. *Joſeph Ralphſon*, and he ſo well approved of them as to make them the 20th and 21ſt *Problems* in the ſecond Edition of his *Analyſis Æquationum Univerſalis*.

And becauſe, in finding the Quantity of *Angles*, every *Circle* is ſuppoſed to be divided into 360 *equal Parts*, called *Degrees*; every *Degree* is ſubdivided into 60 Parts, called *Minutes*; and every Minute into 60 Seconds, &c. (See Page 294.)

Therefore 360) 6,2831853 &c. (0,0174532925 &c. is an Arch of the above-found Periphery, equal to the Arch of one Degree.

And 60) 0,0174532925 &c. (0,0002908882 &c. = the *Arch* of one *Minute*.

Then if the given *Arch* (or *Angle*) be leſs than 45 *Degrees*, reduce it into *Minutes*, and multiply thoſe *Minutes* into this conſtant *Multiplicator*, viz. 0,000290882 calling the Product p. And for the *Sine* ſought put a. Then it will be $-aaaa + 12paaa - 195aa - 36ppaa + 240pa = 45pp$.

EXAMPLE.

Let it be required to find the Sine of $19^\circ.\ 13' = 1153'$. Here $0{,}0002908882 \times 1153 = 0{,}3353940946 = p$. And $-a^4 + 4{,}024729a^3 - 199{,}049611aa + 80{,}494583a = 5{,}06201394$.

Let $r + e = a$

$$\text{Then}\left\{\begin{aligned} & rr + 2re + ee = aa \\ & rrr + 3rre + 3ree + eee = aaa \\ & rrrr + 4rrre + 6rree = aaaa \end{aligned}\right.$$

Note, *In this Caſe the firſt* r *may always be taken equal to the firſt Figure in the Product* = p. Viz. *here* r = 0,3 *which being involved as its Powers direct, and thoſe Powers multiplied into the reſpective Co-efficients of the Equation; it will be*

$$\left\{\begin{array}{llll} +\ 24{,}1483 & +\ 80{,}49e & & \\ -\ 17{,}9144 & -\ 119{,}43e & -\ 199{,}03ee & \\ +\ 0{,}1086 & +\ 1{,}08e & +\ 3{,}62ee & \\ -\ 0{,}0081 & -\ 0{,}11e & -\ 0{,}54ee & \end{array}\right\} = 5{,}06201394$$

Viz. $6{,}3344 - 37{,}97e - 195{,}97ee = 5{,}06201$

Hence

Hence $37{,}97ee + 195{,}97ee = 1{,}27239$
And $0{,}193e + ee = 0{,}006492 = D$

THEOREM $\left\{ \dfrac{D}{,193 + e} = e \right.$

Operation, $0{,}193)\ 0{,}006492\ (0{,}029 = e$
$+ e = ,029 \qquad 42$

1. *Divisor* $,21 \qquad 2292$
1998
2. *Divisor* $,222$

First $r = 0{,}3$
$+ e = 0{,}029$
$r + e = 0{,}329 = r$ for a second Operation.

Which being involved and multiplied, &c. as before, will produce these Numbers.

$+ 26{,}48271781 + 80{,}49458e$
$- 21{,}54532894 - 130{,}97464e - 199{,}0496ee$
$+ 0{,}14332578 + 1{,}30692e + 3{,}9724ee$
$- 0{,}01171611 - 0{,}14244e - 0{,}6494ee$

Viz. $5{,}06899854 - 49{,}31558e - 195{,}7266ee = 5{,}06201394$

Hence $49{,}31558e + 195{,}7266ee = ,0069846$; which being divided by $195{,}7266$ the *Co-efficient* of ee, will become $,25196e + ee = ,0000356854 = D$

Then $\left\{ \dfrac{D}{,25196 + e} = e \right.$

Operation. $0{,}25196)\ ,0000356854\ (0{,}0001415 = e$
$+ e = 0{,}00014 \qquad 2520$

1. *Divisor* $0{,}2520 \qquad 104854$

2. *Divisor* $0{,}25210 \qquad 100840$

40140
25210

Last $r = 0{,}329$ &c.
$+ e = 0{,}0001415$

$r + e = a = 0{,}3291415$ being the natural *Sine* of 90°. 13′. As was required.

Thus you may find the *Right Sine* of any *Arch* or *Angle* less than 45 *Degrees*. But

But, if the given *Arch* be greater than 45 *Degrees*, you must take its Complement to 90°. *viz.* substract it from 90 *Degrees*, and reduce the Remainder into *Minutes*, as before. Then *multiply* the Square of those *Minutes* into this constant *Multiplicator*, 0,0000000084616 calling their *Product* p, and putting a = the *Sine* sought, as before. Then will $a^4 + 28a^3 + 195aa + 36paa108pa - 28a = 196 - 81p$.

Example.

Suppose it were required to find the *Sine* of 75°. 32'. or (which is the same Thing) to find the *Co-sine* of 14°. 28'. = 868', whose *Square* 753424 × 0,0000000084616 = 0,06375172518 = p. Hence the *Equation* in *Numbers* will be $aaaa + 28aaa + 197{,}295062aa - 21{,}114814a = 190{,}8361102588$.

$$\text{Let } r - e = a \qquad \text{And } r = 1$$

$$\text{Then} \begin{cases} rr - 2re + ee = aa \\ rrr - 3rre + 3ree = aaa \\ rrrr - 4rrre + 6rree = aaaa \end{cases}$$

Note, *I here take* $r = 1$ *because the Arch is so near to* 90°. *and therefore I make it* $r - e = a$.

$$\text{Then} \left\{ \begin{array}{rrr} -\ 21{,}1148 + & 21{,}11e + & \\ +\ 197{,}2956 - & 394{,}59e + & 197{,}29ee \\ +\ 28{,}0000 - & 84{,}00e + & 84{,}00ee \\ +\ 1{,}0000 - & 4{,}00e + & 6{,}00ee \end{array} \right\} = 190{,}8361$$

$$\textit{Viz. } 205{,}1808 - 461{,}48e + 287{,}29ee = 190{,}8361$$

$$\text{Hence } 461{,}48e - 287{,}29ee = 14{,}3447$$

$$\text{And } 1{,}606e - ee = {,}049930 = D$$

$$\text{THEOREM} \left\{ \frac{D}{1{,}006 - e} = e \right.$$

Operation. 1,606) ,049930 (0,031 = e
— e = 0,031   471

1. *Divisor* 1,57   2830   First r = 1,000
  1575   — e = 0,031

2. *Divisor* 1,575 $r - e$ = 0,969 = r

&c.

for a second *Operation*; which, being involved as before, will produce these following Numbers.

—20

$$\begin{array}{rrr}
-\ 20{,}460254766 + & 21{,}11481e & \\
+\ 185{,}252368710 - & 382{,}35783e + & 197{,}2951ee \\
+\ 25{,}475889852 - & 78{,}87272e + & 81{,}5960ee \\
+\ 0{,}881647759 - & 3{,}63941e + & 5{,}6337ee \\
\hline
\end{array}$$

Viz. $191{,}149651515 - 443{,}75515e + 284{,}5248ee$
$= 190{,}836110259$

Hence it will be $443{,}75515e - 284{,}5248ee = 0{,}313541256$

And $1{,}55963e - ee = {,}0011019821 = D$

Then $\left\{ \dfrac{D}{1{,}55963 - e} = e \right.$

. . . . . .

Operation. $1{,}55963)\ 0{,}0011019821\ (0{,}0007061 = e$

$-e = 0{,}00070$ 109123

1. *Divisor* $1{,}5589$ 1075210

935358

2. *Divisor* $1{,}55893$

1398520

1247144 &c.

Laſt $r = 0{,}969$

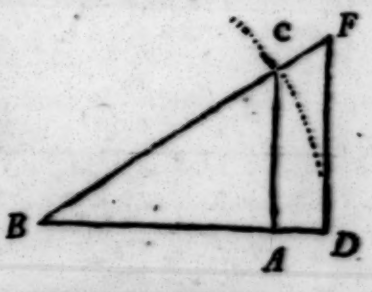
$-\ e = 0{,}0007068$

$r - e = a = 0{,}9682932$ *the Sine* of $75^\circ.\ 32'$. as was required.

Having found the *Sine* and *Co-ſine* of any *Arch*, the *Tangent* is uſually found by this *Proportion*;

Viz. $\left\{\right.$ *As the Co-ſine of any Arch : is to the Sine* of that *Arch* : : *ſo is the Radius : to the Tangent of the ſame Arch.*

For ſuppoſing $BC = BD$ *Radius*, AC the *Sine* of the *Arch* CD. Then BA is the *Co-ſine*, and FD the *Tangent* of the ſame *Arch*. But $BA : CA :: BD : FD$, &c. Now by this *Proportion* there is required to be given both the Sine and *Co-ſine* of the *Arch*, to find the *Tangent*. 'Tis true, if the *Radius*, and either the *Sine* or the *Co-ſine* be given, the other may be found,



thus, $\sqrt{\square BC - \square CA} = BA$. Or $\sqrt{\square BC - \square BA} = CA$. But, if either the *Sine* or *Co-ſine* be given, the *Tangent* may (I preſume) be more eaſily found by the following *Theorems*.

Let

Let $BC=1$. $CA=S$. $BA=x$ and $FD=T$. Then if S be given, T may be found by this

$$\text{Theorem}\left\{\sqrt{\frac{SS}{1-SS}}=T\right.$$

Or if x be given, T may be found by this

$$\text{Theorem}\left\{\sqrt{\frac{xx}{1-xx}}=T\right.$$

Let the Sine of 90°. 13′. (before found) be given, *viz.* 0,3291415 $=S$, to find T the *Tangent* of the ſame *Arch*. Firſt 0,3291415 ×,3291415 = 0,108334127 $=SS$. Again 1 — 0,108334127 = 0,891665873 = 1 — SS. Then 0,891665873) 0,108334127 (0,1214963253 and $\sqrt{}$ 0,1214963253 = 0,3485632 $=T$, the *Tangent*, of 19° 13′. as was required. And ſo you may proceed to find $T=$ the *Tangent*, when $x=$ the *Co-ſine* is given.

Perhaps it may here be expected, that I ſhould have ſhewed and demonſtrated (or at leaſt have inſerted) the Proportions from whence the foregoing *Equations* for making Sines were produced; but I have omitted that, as alſo their Uſe in computing the Sides and Angles of plain Triangles by the Pen only (viz. without the Help of Tables) for the Subject of my Diſcourſe hereafter, if Health and Time permit.

In the mean Time, what is here done may ſuffice to ſhew, that the making of Sines by ſuch a laborious and operoſe Way, as was formerly uſed, is in a great Meaſure overcome; which, I think, I may juſtly claim as my own.

A N

AN INTRODUCTION TO THE MATHEMATICS.

PART IV.

CHAP. I.

Definitions of a Cone, *and its* Section.

THERE are ſeveral Definitions given of a *Cone:* The Learned Dr. *Barrow*, upon *Euclid*, hath it thus:

" A Cone (*ſaith he*) is a Figure made when one Side " of a Rectangle Triangle, (*viz.* one of thoſe Sides that contain " the Right-angle) remaining fixed, the Triangle is turned round " about, 'till it return to the Place from whence it firſt moved: " And if the fixed Right Line be equal to the other which con- " taineth the Right-Angle, then the Cone is a Rectangled Cone; " but if it be leſs, 'tis an Obtuſe-angled Cone; if greater, an " Acute-angled Cone. The *Axis* of a Cone is that fixed Line " about which the Triangle is moved: The *Baſe* of a Cone is the " Circle, which is deſcribed by the Right Line moved about." (*Defin.* 18, 19, 20. *Euclid.* 11.)

Sir *Jonas Moor*, in his Treatiſe of *Conical Sections* (taken out of the Works of *Mydorgius*) defines it thus:

" If a Line of ſuch a Length as ſhall be needful ſhall upon a " Point fixed above the Plane of a Circle, ſo move about the Cir- " cle, until it return to the Point from whence the Motion began, " the Superficies that is made by ſuch a Line is called a *Conical* " *Superficies*; and the ſolid Figure contained within that Super- " ficies and the Circle is called a *Cone*. The Point remaining " ſtill is the Vertex of the Cone, &c."

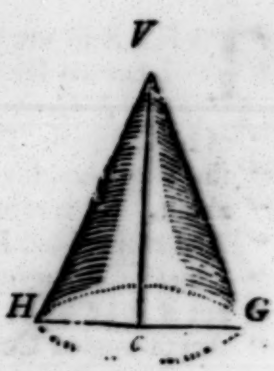
Altho' both these *Definitions* are equally true, and, with a little Consideration, may be pretty easily understood; yet I shall here propose one very different from either of them; and, as I presume, more plain and intelligible, especially to a *Learner*.

If a Circle described upon stiff Paper (*or any other pliable Matter*) of what Bigness you please, be cut into two, three, or more *Sectors*, either equal or unequal, and one of those *Sectors* be so rolled up, as that the *Radii* may exactly meet each other, it will form a *Conical Superficies*.

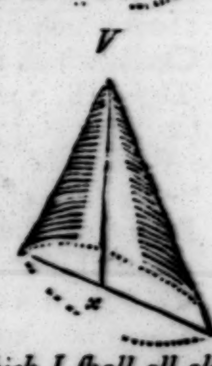
That is, if the *Sector HVG* be cut out of the Circle, and so rolled up as that the *Radii VH* and *VG* way just meet each other in all their Parts, it will form a *Cone*, and the Center *V* will become a *Solid Point*, called the *VERTEX* of the *Cone*; the *Radius VH*, being every where equal, will be the Side of the Cone, and the Arch *HG* will become a Circle, whose *Area* is called the *Cone's Base*.

A *Right Line* being supposed to pass from the *Vertex*, or Point *V*, to the *Center* of the Cone's *Base*, as at *C*, that Line (viz. *VC*) will be the *AXIS*, or *perpendicular Height* of the Cone.

If a *Solid* be actually made in such a Form, it will be a compleat or perfect Cone; which I shall all along call a Right Cone, because its *Axis VC* stands at *Right Angles* with the Plane of its *Base HG*, and its Sides are every where equal.



Any Cone, whose *Axis* is not at *Right Angles* with the Plane of its *Base*, may be properly called an imperfect *Cone*, because its Sides are not every where equal (as in the annexed Figure.) Now, such an imperfect Cone is usually called a *Scalene*, or *Oblique Cone*.



Any solid Cone may be cut by Planes (*which I shall all along hereafter call Right Lines*) into *five Sections*.

Sect.

Sect. 1.

If a Right Cone be cut directly thro' its *Axis*, the Plain or *Superficies* of that *Section* will be a plain Isosceles Triangle, as *HVG Fig.* 2, viz. the Sides *HV* and *VG*) of the Cone's *Base* will be the *Base* of the Triangle, and *(VC)* its *Axis* will be the *perpendicular Height* of the Triangle.

Sect. 2.

If a Right Cone be cut *(any where)* off by a Right Line parallel to its Base, as *b g (it will be easy to conceive, that)* the Plane of that *Section* will be a Circle, because the Cone's Base is such: wherein one Thing ought to be clearly understood, which may be laid down as a *Lemma*, to demonstrate the Properties of the following *Sections*.

LEMMA. *If any two Right Lines, inscribed within a Circle, do cut or cross each other*, as *b g* doth *b b* in the annexed Figure) *the Rectangle made of the Segments of one of the Lines will be equal to the Rectangle made of the Segments of the other Line.* (See *Theorem* 15. *Page* 315.

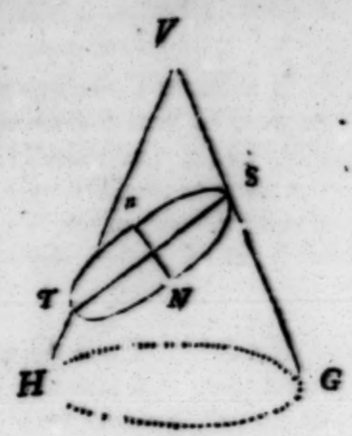
That is, $b\,a \times g\,a = b\,a \times a\,b$
And $H\,A \times G\,A = B\,A \times A\,B$ &c.
consequently if $ba = ab$ and if $BA = AB$ then it will be $b\,a \times g\,a = \Box\, b\,a$, and in the *Cone's* Base $H\,A \times G\,A = \Box\, B\,A$.

Sect. 3.

If a *Right Cone* be (any where) cut off by a *Right Line* that cuts both its Sides, but not parallel to its Base (as *TS* in the following Figure) the Plain of that Section will be an *Ellipsis* (vulgarly called an *Oval)* viz. an oblong or imperfect Circle, which hath several Diameters, and two particular Centers. That is,

1. Any Right Line that divides an *Ellipsis* into two equal Parts is called a Diameter; amongst which the longest and the shortest are particularly distinguished from the rest, as being of most general Use; the other are only applicable to particular Cases.

 2. The



1. The longest Diameter (as *T S*) is called the Transverse Diameter, or Tranverse Axis, being that Right Line which is drawn through the Middle of the *Ellipsis*, and doth shew or limit its Length.

3. The shortest Diameter, call'd the Conjugate Diameter, is a Right Line that doth intersect or cross the Transverse Diameter at Right Angles, in the Middle or common Center of the *Ellipsis* (as *N n*) and doth limit the *Ellipsis's Breadth*.

4. The two Points, which I call particular Centers of an *Ellipsis* (for a Reason which shall be shewed farther on) are two Points in the Transverse Diameter, at an equal Distance each Way from the Conjugate Diameter, and are usually called Nodes, Foci, or burning Points.

5. All Right Lines within the *Ellipsis* that are parallel to one another, and can be divided into two equal Parts, are called Ordinates with Respect to that Diameter which divides them; And if they are parallel to the Conjugate, viz. at Right Angles with the Transverse Diameter, then they are called *Ordinates* rightly applied. And those two that pass through the *Foci* are remarkable above the rest, which, being equal and situated alike, are called both by one Name, *viz.* Latus Rectum, or Right Parameter, by which all the other *Ordinates* are regulated and valued; as will appear farther on.

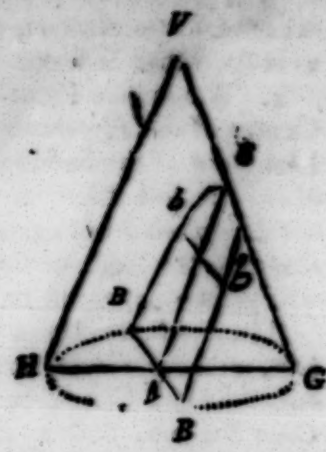
Sect. 4.

If any Cone be cut into two Parts by a Right Line parallel to one of its Sides (as *S A* in the following Scheme) the Plane of that Section (viz. *S b B A B b S*) is called a Parabola.

1. A Right Line being drawn thro' the Middle of any *Parabola* (as *S A*) is called its *Axis*, or intercepted Diameter.

2. All Right Lines that intersect or cut the *Axis* at Right Angles (as *B B* and *b b* are supposed to cut or cross *S A*) are called *Ordinates* rightly applied (as in the *Ellipsis*) and the greatest *Ordinate*, as *B B*, which limits the Length of the *Parabola's Axis* (*S A*) is usually called the Base of the *Parabola*.

3. That



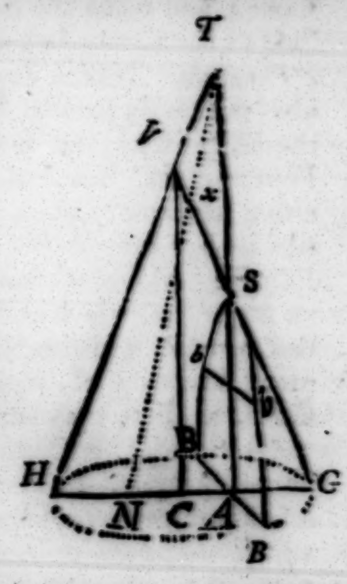
3. That *Ordinate* which passes thro' the *Focus*, or burning Point of the *Parabola*, is called the *Latus Rectum*, or *Right Parameter* (as in the Ellipsis) because by it all the other *Ordinates* are proportioned, and may be found.

4. The *Node*, *Focus*, or burning Point of the *Parabola*, is a Point in its Axis, (but not a Center, as in the *Ellipsis*) distant from the Vertex, or Top of the Section, (viz. from *S*) just ¼ Part of the *Latus Rectum*; as shall be shewn farther on.

5. All Right Lines drawn within a *Parabola* parallel to its *Axis* are called *Diameters*; and every Right Line, that any of those Diameters doth bisect or cut into two equal Parts, is said to be an *Ordinate* to the Diameter which bisects it.

Sect. 5.

If a Cone be any where cut by a Right Line, either parallel to its *Axis*, (as *S A*, or otherwise as *x N*) so as the cutting Line being continued thro' one Side of the Cone (as at *S* or *x*) will meet with the other Side of the Cone if it be continued or produced beyond the Vertex *V*, as at *T*; then the Plane of that Section (viz. the Figure *S b B B b S*) is call'd an HYPERBOLA.



1. A Right Line being drawn thro' the Middle of any *Hyperbola*, viz. within the Section, (as *S A*, or *x* N) is called the *Axis* or intercepted Diameter (as in the Parabola) and that Part of it which is continued or produced out of the Section, until it meet with the other Side of the Cone continued, viz. *T S* or *T x*, &c. is called the *Transverse Diameter*, or *Transverse Axis* of the *Hyperbola*.

2. All Right Lines that are drawn within an *Hyperbola*, at Right Angles to its Axis, are called *Ordinates* rightly applied; as in the *Ellipsis* and *Parabola*.

3 That

3. That Ordinate which passes thro' the *Focus* of the *Hyperbola* is called *Latus Rectum*, or Right Parameter, for the same Reason as in the other Sections.

4. The middle Point of the *Transverse Diameter* is called the Center of the *Hyperbola*; from whence may be drawn two Right Lines (out of the Section) called ASYMPTOTES, because they will always incline (that is, come nearer and nearer) to both Sides of the *Hyperbola*, but never meet with (or touch) them, altho' both they and the Sides of the *Hyperbola* were infinitely extended; as will plainly appear in its proper Place.

These five Sections, *viz.* the *Triangle*, *Circle*, *Ellipsis*, *Parabola*, and *Hyperbola*, are all the Plains that can possibly be produced from a Cone; but of them, the three last are only called *Conic Sections*, both by the antient and modern Geometers.

Scholium.

Besides the foregoing Definitions, it may not be amiss to add, by Way of Observation, how one Section may (or rather doth) change or degenerate into another.

An *Ellipsis* being that Plane of any Section of the Cone which is between the Circle and Parabola, 'twill be easy to conceive that there may be great Variety of Ellipses produced from the same Cone; and when the Section comes to be exactly parallel to one Side of the Cone, then doth the Ellipsis change or degenerate into a *Parabola*. Now a *Parabola*, being that Section whose Plane is always exactly parallel to the Side of the Cone, cannot vary, as the *Ellipsis* may; for so soon as ever it begins to move out of that Position, (viz. *from being parallel to the Cone's Side*) it degenerates either into an *Ellipsis*, or into an *Hyperbola*: That is, if the Section incline towards the Plane of the Cone's Base, it becomes an *Ellipsis*; but if it incline towards the Cone's Vertex, it becomes an *Hyperbola*, which is the Plane of any Section that falls between the *Parabola* and the Triangle. And therefore there may be as many Varieties of *Hyperbola*'s produced from one and the same Cone, as there may be *Ellipses*.

To be brief, a Circle may change into an *Ellipsis*, the *Ellipsis* into a *Parabola*, the *Parabola* into an *Hyperbola*, and the *Hyperbola* into a plain *Isosceles Triangle*: And the Center of the Circle, which is its *Focus* or burning Point, doth, as it were, part or divide itself into two *Foci* so soon as ever the Circle begins to degenerate into an *Ellipsis*; but when the *Ellipsis* changes into a *Parabola*, one End of it flies open, and one of its *Foci* vanishes, and the remaining *Focus* goes along with the *Parabola* when

when it degenerates into an *Hyperbola*: And when the *Hyperbola* degenerates into a plain *Isosceles Triangle*, this *Focus* becomes the vertical Point of the Triangle (viz. the Vertex of the Cone); so that the Center of the Cone's Base may be truly said to pass gradually thro' all the Sections, until it arrives at the *Vertex* of the Cone, still carrying its *Latus Rectum* along with it: For the Diameter of a Circle being that Right Line which passes thro' its Center or Focus, and by which all other Right Lines drawn within the Circle are regulated and valued, may (I presume) be properly called the Circle's *Latus Rectum*: and altho' it loses the Name of Diameter when the Circle degenerates into an *Ellipsis*, yet it retains the Name of *Latus Rectum*, with its first Properties, in all the Sections, gradually shortening as the Focus carries it along from one Section to another, until at last it and the Focus become co-incident, and terminate in the *Vertex* of the Cone.

I have been more particular and fuller in these Definitions than is usual in Books of this Subject, which I hope is no Fault, but will prove of Use, especially to a Learner: And altho' they may perhaps seem a little strange, and at first hard to be understood, yet, when they are well considered, and compared with a Cone cut into such Sections as have been defined, they will not only be found true, but will also help to form a true and clear *Idea* of each Section.

CHAP. II.

Concerning the Chief Properties of an Ellipsis.

Note, *If the transverse Diameter of an* Ellipsis, *as T S in the following* Figure, *be intersected or divided into any two Parts by an* Ordinate rightly applied, *as at the Points A, C, a,* &c. *then are those Parts T A, T C, T a, and S A, S C, S a,* &c. *usually called* Abscissæ *(which signifies Lines or Parts cut off) and by the Rectangle of any two* Abscissæ *is meant the Rectangle of such two Parts as, being added together, will be equal to the Transverse Diameter.*

As $TA + SA = TS$. And $TC + SC = TS$.
Or $Ta + Sa = TS$, &c.

Sect.

Section 1.

Every *Ellipsis* is proportioned, and all such Lines as relate to it are regulated, by the Help of one general *Theorem*.

THEOREM. { *As the Rectangle of any two* Abscissæ: *is to the Square of Half the* Ordinate *which divides them* :: *so is the Rectangle of any other two* Abscissæ: *to the Square of Half that* Ordinate *which divides them.*

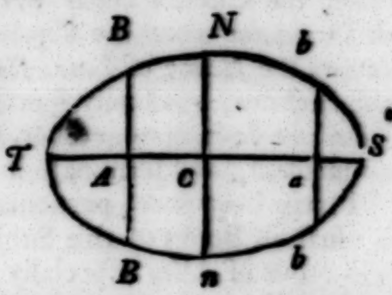
That is,

$TA \times SA : \square BA :: Ta \times Sa : \square ba$

$TA \times SA : \square BA :: TC \times SC : \square NC$

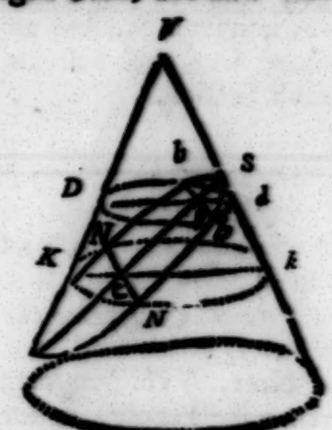
$TC \times SC : \square NC :: Ta \times Sa : \square ba$

&c.



DEMONSTRATION.

Let the annexed Figure represent a Right Cone, cut thro' both Sides by the Right Line TS; then will the Plane of that Section be an *Ellipsis* (by Sect. 3. Chap. 1.) TS will be the Transverse Diameter, NCN and bab will be *Ordinates* rightly applied; as before. Again, if the Lines Dd and Kk be parallel to the Cone's Base, they will be Diameters of Circles (by Sect. 2. Chap. 1.) Then will $\triangle\ TCK$ and $Ta D$ be alike. Also $\triangle\ Sad$ and $\triangle\ SCk$ will be alike.

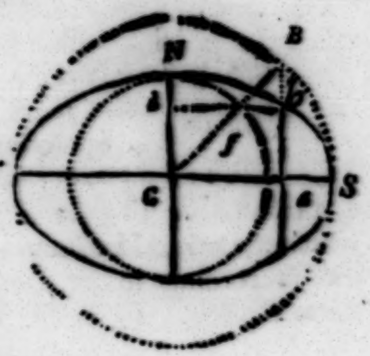


Ergo	1	$Sa : ad :: SC : Ck$	per *Theorem* 13.
And	2	$TC : CK :: Ta : aD$	per *Theorem* 13.
1 ∵	3	$Sa \times Ck = ad \times SC$	
2 ∴	4	$Ta \times CK = TC \times aD$	
2 × 3	5	$Sa \times Ck \times Ta \times CK = ad \times SC \times TC \times aD$.	Per *Axiom* 3.
But	6	$CK \times Ck = \square NC$	per *Lemma Sect.* 2.
And	7	$aD \times ad = \square ba$	per *Lemma Sect.* 2.
Then		for $CK \times Ck$, and $aD \times ad$, take $\square NC$ and $\triangle ba$	
5, 6, 7	8	$Sa \times Ta \times \square NC = TC \times SC \times \square ba$	Per *Axiom* 5.
Hence	9	$Sa \times Ta : \square ba :: TC \times SC : \square NC$.	See *Page* 194.

Q. E. D.

Or

Or, the Truth of these Proportions may be otherwise prov'd by a Circle, without the Help of the Cone; thus: Let any *Ellipsis* be circumscribed and inscribed with Circles, as in the following Figure; then from any Point in the circumscribed Circle's Periphery, as at *B*, draw the Right Line *Ba*, parallel to the semi-conjugate Diameter *Nc*, then will *ba* be a *Semi-ordinate* rightly applied to the tranverse Diameter *TS*, as before. Again, from the Poipt *b* (in the *Ellipsis's Periphery)* draw the Right Line *bd* parallel to the Transverse *TS*; and draw the *Radius BC*. Then will $\triangle BCa$ and $\triangle Cfd$ be alike.



* Therefore	1	$BC : Ba :: Cf : dC$ per *Theorem* 12.
But	2	$TC = BC$, $NC = Cf$ and $ba = dC$
Conseq.	3	$TC : Ba :: NC : ba$
Or	4	$TC : NC :: Ba : ba$
4 in □'s	4	$\square TC : \square NC :: \square Ba : \square ba$
But	6	$Ta \times Sa = \square Ba$ per Lem. *Sect*. 2. *Ch*. 1.
Therefore	7	$Ta \times Sa : \square ba :: TC \times SC = \square TC : \square NC$, as before.

And so for any other *Abscissæ* and their *Semi-ordinates*.

These Proportions being found to be the true and common Properties of every *Ellipsis*, all that is farther required in (or about) that Section may be easily deduced from them.

Sect. 2. *To find the* Latus Rectum, *or* Right Parameter *of any Ellipsis.*

THERE are several Ways of finding the *Latus Rectum*, but I think none so easy, and shews it so plainly to be the Third principal Line in the *Ellipsis*, as the following.

Theorem. *As the* Transverse Diameter : *is in Proportion to the* Conjugate : : *so is the* Conjugate : *to the* Latus Rectum.

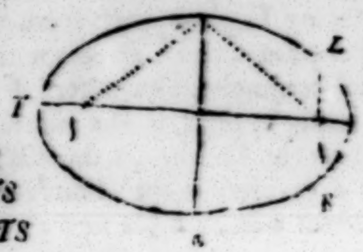
Viz. *(in the following Fig.)* $TS : Nn :: Nn : DR$ the *Latus Rectum*.

Demonstration.

From the last *Proportions* take either of the Antecedents, and its Consequent, *viz*, either $TC \times SC : \square NC$, or $Ta \times Sa : \square ba$,

 and

and make TS the third Term, to which find a fourth Proportional, and it will be $= LR$:



Thus	1	$TC \times SC : \square NC :: TS : LR$
But	2	$\begin{cases} TC = SC \\ \text{and } NC = Cn \end{cases}$
Therefore	3	$TC \times SC = \frac{1}{4} \square TS$
And	4	$\square NC = \frac{1}{4} \square Nn$
1, 3, 4	5	$\frac{1}{4} \square TS : \frac{1}{4} \square Nn :: TS : LR$
5 ∵	6	$\frac{1}{4} \square TS \times LR = \frac{1}{4} \square Nn \times TS$
$6 \times \overline{4}$	7	$\square TS \times LR = \frac{1}{4} \square Nn \times TS$
$7 \div TS$	8	$TS \times LR = \square Nn$ which gives the following *Analogy*.
viz.	9	$TS : Nn :: Nn : LR$
Again	10	$TC \times SC : \square NC :: Ta \times Sa : \square ba$ by common *Properties*.
1, 10	11	$TS : LR :: Ta \times Sa : \square ba$,

From hence 'tis evident that LR, thus found, is that *Ordinate* by which the other *Ordinates* may be regulated and found. Therefore (according to its Definition *Sect.* 3. *Chap.* 1.) it is the true *Latus Rectum*. Q. E. D.

Consectary.

Hence it follows, that if the transverse and conjugate Diameters of any *Ellipsis* are given (either in Lines or Numbers) the *Latus Rectum* may be easily found; and then any *Ordinate*, whose Distance from the Conjugate is given, may be found, as above.

Sect. 3. *To find the* Focus *of any* Ellipsis.

THE *Focus* is the Distance of the *Latus Rectum* from the Conjugate or Middle of the *Ellipsis* (vide Definition 4, Page 364.) and that Distance is always a mean *Proportional* between the half Sum and half Difference of the Transverse and conjugate Diameters, which gives this *Theorem*.

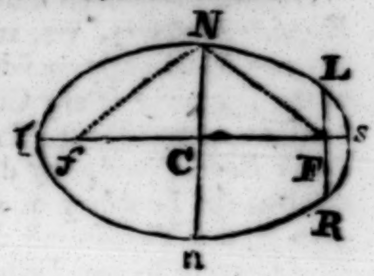
THEOREM. *From the* Square *of half the* Tranverse *substract the* Square *of half the* Conjugate, *the square Root of their Difference will be the Distance of each* Focus *from the Middle or common Center of the Ellipsis.*

That is, supposing the Points f and F to be the two *Foci*, viz. $fC = CF$, and $TC = \frac{1}{2} TS$. $NC = \frac{1}{2} Nn$. Then, $TC + NC : fC :: FC : TC - NC$. Ergo $\square FC = \square TC - \square NC$. Consequently, $FC = \sqrt{\square TC - \square NC}$.

DEMON-

Demonstration.

First,	1	$TS \times LR = \square Nn$, by 8th Step of the last Process.
And	2	$TS : LR :: TF \times SF : \square LF$, *Per.* common Properties.
That is,	3	$TS : LR :: \overline{TC + CF} \times \overline{TC - CF} : \frac{1}{4}\square LR = \square LF$
$3 \because$	4	$\left\{ \frac{1}{4}\square LR \times TS = \overline{\square TC - \square CF} \times LR \right.$
$4 \div LR$	5	$\frac{1}{4} LR \times TS = \square TC - \square CF$
$1 \div \overline{4}$	6	$\frac{1}{4} TS \times LR = \frac{1}{4}\square Nn = \square NC$
5, 6	7	$\square NC = \square TC - \square CF$
7, $\overline{+}$	8	$\square CF = \square TC - \square NC$
$8\ \omega\omega^{\frac{1}{2}}$	9	$CF = \sqrt{\square TC - \square NC}$



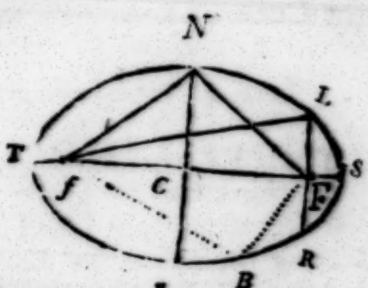
Now from hence is deduced that notable Proposition, upon which is grounded the usual Method of describing an *Ellipsis*, and drawing of *Tangents*, &c.

Proposition. *If from the two* Foci *of any* Ellipsis *there be drawn two* Right Lines, *so as to meet each other in any Point of the* Ellipsis's Periphery, *the Sum of those Lines will be equal to the* Transverse.

Viz. $fN \times NP = TS$. $fL + LF = TS$. Or $fB + BE = TS$, &c.

Demonstration.

First,	1	$\left\{ \square CF + \square NC = \square TC \right.$ *by 8th of the last.*
But	2	$\left\{ \square CF + \square NC = \square NF \right.$ *by* Theorem 11.
1, 2	3	$\left\{ \square NF = \square TC \right.$ *by* Axiom 5.
$3\ \omega\omega^{2}$	4	$NF = TC$
Hence		$2NF = 2TC = TS$
Again,	5	$TS : LR :: TF \times FS : \square LF$, by common Properties.
Conseq.	6	$\frac{1}{2} TS : \frac{1}{2} LR :: TF \times FS : \square LF$
But		$\frac{1}{2} TS = TC$. And $\frac{1}{2} LR = LF$
Ergo	7	$TC : LF :: TC + CF \times TC - CF : \square LF$
$7 \because$	8	$TC + LF = \square TC - \square CF$
But,	9	$\square fF + \square LF = \square fL$, *by* Theorem 11.
That is,	10	$4\square CF + \square LF = \square fL$, for $2CF = fF$
$8 \times \overline{4}$	11	$4\square TC - 4\square CF = 4TC \times LF$
10 + 11	12	$4\square TC + \square LF = 4TC \times LF + \square fL$
12 −−	13	$4\square TC - 4TC \times LF + \square LF = \square fL$



$13 ww^2$	14	$2TC - LF = fL$
$14 + LF$	15	$2TC = fL + LF$. But $2TC = TS$
Ergo	16	$TS = fL + LF$. Q. E. D.

And the Proposition must needs hold true to every Point in the *Ellipsis's Periphery*, viz. at *B*, &c. As will evidently appear to any one who rightly considers, That, as a Thread just the Length of the Diameter of any Circle having its two Ends tied together, and then moved about a Point in the Center (viz. *by making it a double Radius*) will, by drawing another Point in its Extremity, describe the *Periphery* of a Circle; [*vide* Definition, Page 280] even so, if a Thread just the Length of the transverse Diameter (*T S*) having its two Ends so fixed upon the two *Foci* (*f* and *F*) that it may be moved about them, by drawing a Point in its Extremity (viz. *at its full Stretch*) it will describe the true *Periphery* of an *Ellipsis*.

Now, altho' this easy Way of describing, or, as usually phras'd, drawing an *Ellipsis*, be mechanical, and known even to most *Joiners*, *Carpenters*, &c. yet it gives as compleat and clear an Idea of that Figure, as any other Way whatsoever; and by describing it thus about its two *Foci*, as a Circle is about its Center, doth plainly shew that those two Points are not improperly called particular Centers in Definition 4, Sect. 3, Chap. 1. for each of them bears much the same Respect to the *Ellipsis*'s Periphery, as the Circle's Center doth to its Periphery.

Sect. 4. *To describe or delineate an* Ellipsis *several Ways.*

THERE are several (other) Ways of describing an *Ellipsis*, both Geometrically and Numerically, according to peculiar Occasions, but I shall only mention two or three of them, leaving the rest to the *Learner*'s Genius. Now, in order to that Work, it will be convenient to consider what Lines are requisite to limit or bound its Form, which I take to be chiefly these following.

1. If the Transverse and Conjugate are given, the *Ellipsis* is perfectly limited; (vide *Consectary Page* 370.) for if *T S* and *N n* be set at *Right Angles* in their Middle at *C*, and *T C* or *C S* be set *off* from *N*, or *n*, both Ways upon the Transverse to *f* and *F*, (viz. make $fN = TC = NF$) then will those Points *f* and *F* be the two *Foci* (by 4th Step of the last Process) and then the *Ellipsis* may be described as above.

2. If

2. If the Transverse Diameter and Latus Rectum are given, the Ellipsis is truly limited, because by them the Conjugate may be found, by Sect. 2.

3. Or if only the Traesverse, and the Proportion it hath either to the Conjugate or Latus Rectum, be given, the *Ellipsis* is thereby limited. As for Instance; suppose the given Ratio between the Transverse and Conjugate to be, as a : to d:

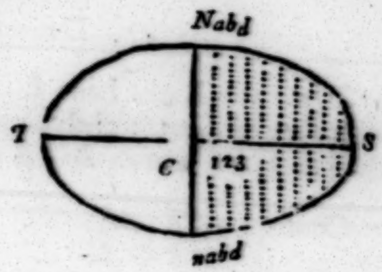
Viz. $a:d::TS:Nn$, then $\frac{TS \times d}{a} = Nn$, &c.

4. If either the Transverse or Conjugate, and the Distance of the Focus from the Conjugate be given, the Ellipsis is limited, because by them the Conjugate or Transverse may be found.

These being premised, and the precedent Work a little considered, it must be easy to describe or delineate any Ellipsis in *Plano*, either geometrically or numerically.

1. *To describe an* Ellipsis *numerically by Points.*

Suppose the Transverse Diameter $TS = 20$, and the Conjugate $Nn = 12$, (*either Inches, or any other equal Parts*) and let them cross each other at *Right Angles* in their Middles, as in the Point C; then will $TC = CS = 10$, and $NC = Cn = 6$, and it will be $20 : 12 :: 12 : 7,2 =$ the Latus Rectum.



Again $20 : \overline{7,2}$. Or $\overline{\text{rather}}$ take their *Ratio*.

Thus $\begin{cases} 1 : 0,36 :: \overline{10+1} \times \overline{10-1} : \square a. \| 1. \\ 1 : 0,36 :: \overline{10+2} \times \overline{10-2} : \square b. \| 2. \\ 1 : 0,36 :: \overline{10+3} \times \overline{10-3} : \square d. \| 3. \text{ \&c.} \end{cases}$

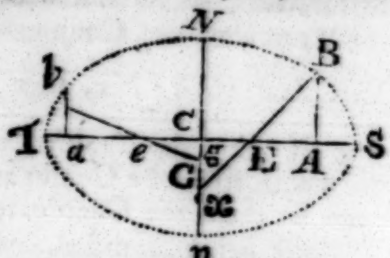
Viz. $\begin{cases} \overline{100-1} \times 0,36 = \square a.1. & \text{Hence } \sqrt{\overline{99 \times 0,36}} = 5,97 \text{ \&c.} = a.\ 1 \\ \overline{100-4} \times 0,36 = \square b.2. & \sqrt{\overline{96 \times 0,36}} = 5,88 \text{ \&c.} = b.2 \\ \overline{100-9} \times 6,36 = \square d.3. & \sqrt{\overline{91 \times 0,36}} = 5,72 \text{ \&c.} = d.3 \end{cases}$

If so many Semi-ordinates as may be thought convenient (the more the better) be found in this Manner, and every one of them be set off at *Right Angles* from its respective Point in the Transverse Diameter each Way, viz. from 1 to a, from 2 to b, from 3 to d, &c. Then if a Curve Line be carefully drawn with an even Hand thro' those extreme Points a, b, d, &c. it will be the Ellipsis's Periphery required.

2. To

2. *To describe an* Ellipsis *Geometrically by Points.*

Having the Transverse and Conjugate Diameters given, *viz.* TS and Nn, placed at *Right Angles* in their Middlss, as before: Then from either End of the Conjugate, *viz.* N (or n) set off half the Transverse Diameter to x. That is, make $Nx = TC$ *(continuing the* Conjugate Nn *when it is shorter than* TC*)* Or, which is all one, make $Cx = TC - NC$. Then take any Point in the Line Cx at Pleasure; suppose it at G, and from that Point at G set off the Distance Cx to the Transverse (as at E) *viz.* make $GE = Cx$, and join the Points GE with a Right Line, produced so far beyond E as to make $EB = MC$. Consequently $GB = TC$.



Then, I say, where-ever the Point G was taken between C and z the *Point B* will just touch (or fall in) the Ellipsis's Periphery.

Demonstration.

Draw the Right Line BA perpendicular to TS, *viz.* let BA be a *Semi-ordinate* rightly applied to the transverse Diameters TS; then $\triangle\, GCE$ and $\triangle\, BAE$ will be alike.

Conseq.	1	$CE : AE :: EG : EB$, by Theorem 13.
1, and	2	$CE+AE : AE :: EG+EB : EB$. See *p.* 192.
But	3	$CE+AE = CA$. $EG+EB = TC$. And $EB = NC$
Therefore	4	$CA : AE :: TC : NC$
6, in □'s	5	$\square CA : \square AE :: \square TC : \square NC$
5, ∵	6	$\frac{\square CA \times \square NC}{\square TC} = \square AE$
But	7	$\square NC - \square AB = \square AE$
That is,		$\square EB - \square AB = \square AE$
6, 7	8	$\frac{\square CA \times \square NC}{\square TC} = \square NC - \square AB$
8 × □TC	9	$\square CA \times \square NC = \square NC \times \square TC - \square AB \times \square TC$
9 +	10	$\square NC \times \square TC - \square CA \times \square NC = \square AB \times \square TC$
10, *Analogy*	11	$\square TC : \square NC :: \square TC - \square CA : \square AB$
That is,	12	$TC \times CS : \square NC :: \overline{TC+CA} \times \overline{TC-CA} : \square AB$

which is according to the common *Properties* of the *Ellipsis*: Therefore the Point B is truly found.

Q. E. D.

Hence

Hence it follows, that if a convenient Number of such Lines as *G E B* be so drawn (as above directed) from the like Number of Points taken between *C* and *x*, &c. their extream Points (as at *B*) will be those Points by which (with an even Hand) the *Ellipsis* may be truly described, as before.

But, if this be well understood, it will be very easy to conceive how to describe an *Ellipsis* very readily, without drawing those Lines, by having a thin, streight, narrow Ruler just the Length of *T C*, made somewhat sharp at both Ends, upon which, from one of its Ends, set off the Length of *N C*. Then, if the Point upon the Ruler which represents *E* be gradually or easily moved along the Transverse *T S*, and at the same time the Point or End representing *G* be kept sliding close along the Conjugate *N n*, 'tis evident from the Work above, that the End of the Ruler representing *B* will, by that Motion, assign the true *Periphery* of the Ellipsis required; for by that Motion the streight Edge of the Ruler doth supply an infinite Number of the aforesaid Lines; as will appear very plain and easy in Practice.

Scholium.

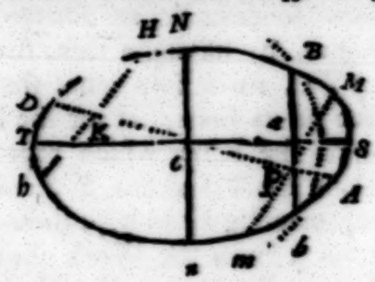
Now from hence was deduced the first Invention of that well-contrived Instrument for drawing an Ellipsis by one Motion, commonly called the Elliptical Compasses, being usually made of Brass, and composed of three Parts, two of which represent (or rather supply) the transverse and conjugate Diameters set together at *Right Angles*; and the third part is a moveable Ruler, which performs the Office of the last mentioned thin Ruler. But because the making of it is so well known to most Mathematical Instrument-makers, especially to that accurate and ingenious Artist Mr. JOHN ROWLEY, Mathematical Instrument-maker, under St. *Dunstan*'s Church in *Fleet-street*, *London*; who, for his great Skill in contriving, framing, and graduating all Kind of Mathematical Instruments, may, I believe, be justly called one of the best Workmen of his Trade in *Europe*; I think it needless therefore to give a particular Description of that Instrument.

Also from hence came that ingenious Invention of making Engines for turning all Sorts of elliptical or oval Work, as oval Boxes, Picture-frames, &c.

Sect.

Sect. 5. *Any Ellipsis being given, to find its* TRANSVERSE *and* CONJUGATE *Diameters.*

Suppose the given *Ellipsis* to be *T N S n* (in the annexed Scheme) in which let it be required to find the Transverse Diameter *T S* and its Conjugate *N n*. Draw within the *Ellipsis* any two *Right Lines* parallel to each other as *H b* and *M m*, and bisect those Lines, *viz.* find the Middle Point of each, as at *K* and *P*; then thro' those Points *K* and *P* draw a Right Line, as *D A*, and it will be a Diameter; for it will divide the Ellipsis into two equal Parts, [See Defin. 1, Page 363.] consequently the Middle of *D A* will be the true Middle or common Center of the Ellipsis, as at *C*.



For 'tis the Nature and Property of all Diameters, *howsoever they are drawn in an* Ellipsis (*as it is in a* Circle) *to cut or cross one another in the common Center or Middle of the Figure, as at* C.

Upon the Point *C* describe an *Arch* of any Circle that will cut the Ellipsis's Periphery in two Points, as at *B* and *b*; then join these Points *B b* with a *Right Line*, and it will be an *Ordinate*, thro' whose Middle (as at *a*) and the common Center *C*, the tranverse Diameter *T S* must pass. For *B S* = *S b*, and *B a* is at Right Angles with *T S*; therefore the Line *B b* is an *Ordinate* rightly applied to *T S* the transverse Diameter. And if thro' the Point *C* there be drawn the Right Line *N n* parallel to *B b*, it will become the Conjugate; as was required.

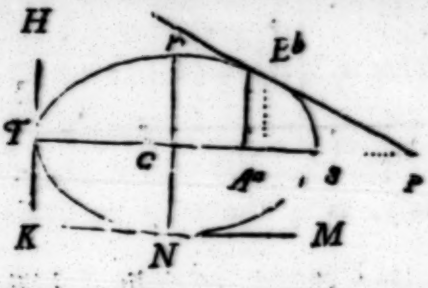
Sect. 6. *To draw a* TANGENT, *or Right Line that may touch the Ellipsis's Periphery in any assigned Point.*

The Drawing of Tangents to or from any assigned Point in the Ellipsis's Periphery, admits of three Cases.

Case 1. If it be required to draw a Tangent that may touch the Ellipsis in either of the extream Points of its transverse Diameter, as at *T* or *S*, it is plain the Tangent must be drawn parallel to the conjugate Diameter *N n*; as *H K* in the following Figure is supposed to be.

Case

Case 2. Or, if the *Tangent* must be drawn to touch the *Ellipsis* in either of the extream Points of its *Conjugate Diameter*, as at *N* or *n*, it is as evident that it must be drawn parallel to the *Transverse Diameter T S*, as *K M*. Consequently if that *Tangent* and the *Transverse* were both infinitely continued, they would never meet.



Case 3. But if it be required to draw a *Tangent* that may touch the *Ellipsis* in any other *Point*, as at *B*, &c. Then, if the *Tangent* and the *Transverse Diameter* be both continu'd, they will meet in some *Point*, as at *P*; and those two Points (viz. *B* and *P*) do so mutually depend upon each other, that one of them must be assigned in order to find the other, that so the *Tangent* may by them be truly drawn. Let $D = TS$. $y = AS$, and $z = AP$. Then, if y be given, z may be found by this

Theorem $\left\{\dfrac{Dy - yy}{\frac{1}{2}D - y} = z\right.$. Or, if z be given, y may be found by

this *Theorem* $\left\{\dfrac{D+z}{2} \pm \sqrt{\left\{\dfrac{DD+zz}{4}\right.}\right. = y$

Demonstration.

Draw the *Semi-ordinate ba*, as in the *Figure*; then will $\triangle BAP$ and $\triangle baP$ be alike. Put $x = Aa$ the *Distance* between the two *Semi-ordinates* (viz. between *BA* and *ba*) which we suppose infinitely small.

Then	1	$z : z - x :: BA : ba$, by *Theorem* 13.
But	2	$D - y \times y : D - y + x \times y - x :: \square BA : \square ba$
That is,	3	$Dy - yy : Dy - yy + yx - Dx - xx :: \square BA : \square ba$
1 in □'s,	4	$zz : zz - 2zx + xx :: \square BA : \square ba$
Suppose	5	$x = 0$, that so x may be every where rejected.
3, Then	6	$Dy - yy : Dy - yy + 2y - D :: \square BA : \square ba$
4, And	7	$zz : zz - 2z :: \square BA : \square ba$
6, 7	8	$Dy - yy : Dy - yy + 2y - D :: zz : zz - 2z$
8 ∵	9	$2yzz - Dzz = 2yyz - 2Dyz$
9 ÷ 2z	10	$yz - \frac{1}{2}Dz = yy - Dy$
10 $\pm$	11	$\frac{1}{2}Dz - yz = Dy - yy$

11 ÷	12	$z = \dfrac{Dy - yy}{\frac{1}{2}D - y}$	which is the 1st *Theorem*, and gives the following *Analogy*.
Analogy	13	$\frac{1}{2}D - y : y :: D - y : z$.	Viz. $CA : SA : TA : AP$
10 − yz	14	$yy - Dy - yz = -\frac{1}{2}Dz$	
14 $C\square$	15	$yy - Dy - yz + \frac{1}{4}\overline{D - D\frac{1}{2}}Dz + \frac{1}{4}zz = \frac{1}{4}DD + \frac{1}{4}zz$	
15 w^2	16	$y - \frac{1}{2}D - \frac{1}{2}z = \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$	
That is,	17	$y = \frac{1}{2}D + \frac{1}{2}z \pm \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$	which is the 2d *Theorem*. Q. E. D.

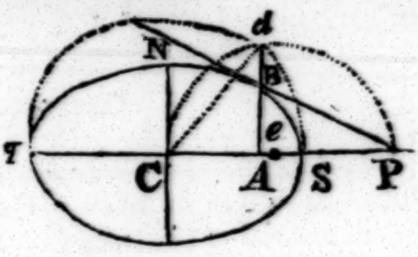
The *Geometrical* Performance of these two *Theorems* is very easy, as by the following *Figure*.

1. Suppose the *Point B* in the *Ellipsis Periphery* were given, and it were requir'd to find the *Point P*, &c.

Make *T C Radius*, and upon the common Center *C* describe the *Semi-circle T d S*, and join the *Points C* and *d* with a *Right Line*; then bisect that *Line* (by *Prob.* 2, *Pag.* 287) and mark the *Point* where the bisecting *Line* would cross the *Transverse*, as at ‖ *e*. Upon that *Point* ‖ *e*, with the *Radius Ce (or Cd)* describe another *Semi-circle*, producing the *Transverse Diameter* to its *Periphery*, and it will assign the *Point P*.

For if $D = \overline{TS}$, $y = AS$, $z = AP$, as before.

Then	1	$\overline{D - y} \times \underline{y} = \square dA$
And	2	$\overline{\frac{1}{2}D - y} \times z = \square dA$
For	3	$TA : dA :: dA : SA$
And	4	$CA : dA :: dA : AP$
But	5	$CA = \frac{1}{2}D - y$, &c.
1, 2	6	$\frac{1}{2}Dz - yz = Dy - yy$ as at the 11th *Step* before.



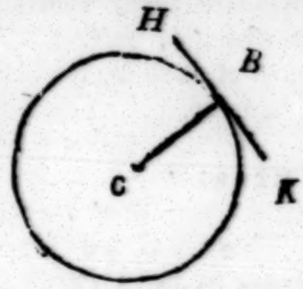
Therefore the *Point P* is truly found. Consequently, if a *Right Line* be drawn through those Points *B* and *P*, it will be the *Tangent* required, according to the first *Theorem*.

2. The *Converse* of this is as easy, to wit, if the *Point P* be given, thence to find the *Point B* in the *Ellipsis Periphery*. Thus, circumscribe half the *Ellipsis* with the *Semi-circle T d S*, as before; and bisect the Distance between the *Points C* and *P*, as at *e*, *viz.* Let $C\,e = e\,P$. Then making *C e Radius*, upon the Point *e*, describe the *Semi-circle C d P*; and from the *Point* where the two *Semi-circles* intersect or cross each other, as at *d*, draw the *Right Line d A* perpendicular to the *Transverse T S*,

T S, and it will assign the Point of Contact *B* in the *Ellipsis Periphery* through which the *Tangent* must pass.

But the *practical Method* of drawing *Tangents* to any assign'd Point in the *Ellipsis Periphery* may (*without finding the aforesaid Point P*) be easily deduced from the following *Property* of *Tangents* drawn to a *Circle*, which is this.

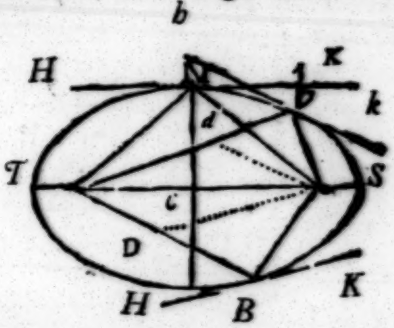
If to any *Radius* of a *Circle*, as *CB*, there be drawn a *Tangent Line* (*as HK*) to *touch* the *Radius* at the *Point B*; the two *Angles*, which the *Tangent* makes with the *Radius*, will always be two *Right Angles* (16, 17, 18, 19 *Euclid* 3.) that is, $< H B C = < C B K = 90°$.



In like Manner the two *Angles*, made between the *Tangent* and the two Lines drawn from the *Foci* of any *Ellipsis* to the *Point* of *Contact*, will always be equal, but not *Right Angles*, save only at the two Ends of the *Transverse Diameter*.

These being well consider'd, and compared with what hath been said in *page* 372, it must needs be easy to understand the following Way of drawing *Tangents* to any *assign'd Point* in the *Ellipsis Periphery*; which is thus:

Having by the *transverse* and *conjugate Diameters* found the two *Foci f* and *F*, by *Sect.* 3. from them draw two *Right Lines* to meet each other in the *assign'd Point* of *Contact*, as *f b* and *F b* (or *f B* and *F B*) in the *annex'd Figure*. Next set off (viz. make) $b\,d = b\,F$ (or $B\,D = B\,F$) and join the *Points F d* (or *F D*) with a *Right Line*.



Then, I say, if a *Right Line* be drawn through the *Point* of *Contact b* (or *B*) parallel to *d F*, or *D F*, it will be the *Tangent* requir'd. For it is plain, that as the $< f N H = < F N K$ when the *Tangent* is parallel to the *Transverse Diameter*, even so is the $< f b b = < F B k$, (and $< f B H = < F B K$) and will be every where so, as the *Point* of *Contact b* (or *B*) and its *Tangent* is carried about the *Ellipsis Periphery* with the *Lines f b F* (or *f B F*)

CHAP. III.

Concerning the Chief Properties of every PARABOLA.

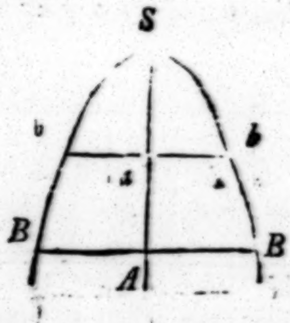
NOTE, in every Parabola, the intercepted Diameter, or that Part of its Axis, which is between the Vertex and that Ordinate which limits its Length, as Sa or SA, &c. is call'd an Abſciſſa, *Sect.* 1. The Plan or Figure of every Parabola is proportioned by its Ordinates and Abſciſſæ, as in the following Theorem.

THEOREM { *As any one Abſciſſa : is to the Square of its Semi-ordinate : : ſo is any other Abſciſſa : to the Square of its Semi-ordinate.*

That is, if we ſuppoſe the *annex'd Figure* to be a *Parabola,* wherein Sa, and SA, are *Abſciſſæ,* and bab, BAB, *Ordinates* rightly apply'd, it will

be $Sa : \square ba :: SA : \square BA$ { whereſoever the *Points* a,
or $Sa : SA :: \square ba : \square BA$ {

A, are taken, and ſo for any other *Abſciſſæ, &c.*

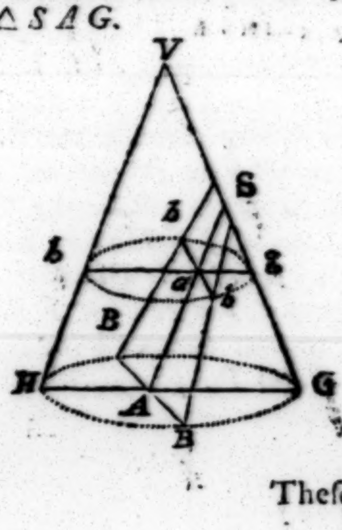


DEMONSTRATION.

Let the following Figure HVG repreſent a *Right Cone* cut into two Parts by the *Right Line* SA, parallel to its Side VH. Then the *Plain* of that *Section*, viz. $BbSbB$ will be a *Parabola,* by *Sect.* 4. *page* 364, wherein let us ſuppoſe SA to be its *Axis,* and bab, BAB to be *Ordinates* rightly apply'd to that *Axis.* Again, imagine the *Cone* to be cut by the *Right Line* bg parallel to its *Baſe* HG. Then will bg be the *Diameter* of a *Circle*, by *Sect.* 2. *page* 363. and $\triangle Sag$ like to $\triangle SAG$.

Therefore	1	$Sa : ag :: SA : AG$ By *Theorem* 13.
1 ∵	2	$Sa \times AG = SA \times ag$
$2 \times ba$	3	$SA \times AG \times ba = SA \times ag \times ba$ By *Axiom* 3.
But	4	$HA = ba$, becauſe SA is parallel to VH
And	5	$\square BA = AG \times HA$; $\square ba = ag \times ba$ By Lem. P. 363
3, 4, 5,	6	$SA \times \square BA = SA \times \square ba$ By *Axiom* 5.
6, *Analogy*	7	$Sa : \square ba :: SA : \square BA$ Vide *Page* 194.

Q. E. D.



These

These *Proportions* being prov'd to be the common *Property* of every *Parabola*, all that is farther requir'd about that *Section*, or *Figure*, may from thence easily be deduced.

Sect. 2. *To find the* Latus Rectum *or Right* Parameter *of any Parabola.*

The *Latus Rectum* of a *Parabola* hath the same *Ratio* or *Proportion* to any *Abscissa*, and its *Semi Ordinate*, as the *Latus Rectum* of any *Ellipsis* hath to its *Transverse* and *Conjugate Diameters*, and may be found by this Theorem.

Theorem $\left\{\right.$ *As any Abscissa : is in Proportion to its Semi-ordinate :: so is that Semi-ordinate : to the* Latus Rectum.

Let $L =$ the *Latus Rectum*.

Then	1	$Sa : ba :: ba : L$ $\Big\}$ where-ever the *Points* a, and A, are taken in the *Axis*.
And	2	$SA : BA :: BA : L$
1 $\because$	3	$\dfrac{\square ba}{Sa} = L$: Or $Sa \times L = \square ba$
1 $\because$	4	$\dfrac{\square BA}{SA} = L$: Or $SA \times L = \square BA$
$3=4$	5	$\dfrac{\square BA}{SA} = \dfrac{\square ba}{Sa}$ Per *Axiom* 5.
$5 \times$	6	$Sa \times \square BA = SA \times \square ba$, which gives this
Analogy	7	$Sa : \square ba :: SA : \square BA$, the same as at the 7th Step of

the last *Process*; therefore L (thus found) is the true *Latus Rectum*, by which all the *Ordinates* may be regulated and found, according to its *Definition* in *Section* 4, *page* 364. For by the third Step $Sa \times L = \square ba$, and by the 4th Step $SA \times L = \square BA$. Consequently $\sqrt{Sa \times L} = ba$ and $\sqrt{SA \times L} = BA$; and so for any other *Ordinate*.

Or if the *Ordinates* are given, to find their *Abscissæ*; then it will be, $L : ba :: ba : Sa$, and $L : BA :: BA : SA$, &c.

Consequently $\dfrac{\square ba}{L} = Sa$, and $\dfrac{\square BA}{L} = SA$, &c.

From the Consideration of these *Proportions*, it will be *easy* to conceive how to find the *Latus Rectum Geometrically*, thus:

Join

Join the *vertical Point* S of the *Axis*, and either *extream Point* of any *Ordinate*, as B (or b) with a *Right Line*, viz. SB (or Sb) and bisect that Line (by *Problem* 2. *page* 287.) marking the *Point* where the bisecting *Line* doth intersect or cross the *Axis*, as at E (or e) and with the *Radius* SE (or Se) upon the *Point* E (or e) describe a *Circle*; (as in the annex'd Figure) then will the Distance between the *Ordinate* and that *Point* where the *Circle's Periphery* cuts the *Axis*, viz. AR (or ar) be the true *Latus Rectum* required.

For $SA : BA :: BA : AR$, and $Sa : ba :: ba : ar$, by *Theor.* 13. therefore $AR = L$. And $ar = L$, by the 1st and 2d *Steps* above.

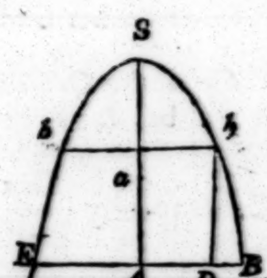
Consectary.

From these *Proportions* of finding the *Latus Rectum*, it will be easy to deduce and demonstrate this following *Theorem.*

THEOREM. { *As the* Latus Rectum : *Is to the Sum of any two Semi-ordinates* :: *so is the Difference of those two Semi-ordinates* : *to the Difference of their Abscissæ.*

Suppose any *Right Line* drawn within the *Parabola*, as bD, parallel to its *Axis* SA; then will that *Line* (viz. bD) be a *Diameter* (by *Def.* 5. *pag.* 365) which will make $ED = AB + ab$, $DB = AB - ab$, and $bD = SA - Sa$. Then it will be $L : ED :: DB : bD$, according to the *Theorem.*

DEMONSTRATION.



First	1	$SA = \frac{\square BA}{L}$, by *Step* 2. of the last *Process.*
And	2	$Sa = \frac{\square ba}{L}$ by *Step* 1. of the last *Process.*
1 — 2	3	$SA - Sa = \frac{\square BA - \square ba}{L}$
3 × L	4	$\overline{SA - Sa} \times L = \square BA - \square ba$
But	5	$\square BA - \square ba = \overline{BA + ba} \times \overline{BA - ba}$
4 = 5	6	$\overline{SA - Sa} \times L = \overline{BA + ba} \times \overline{BA - ba}$
6, *Analogy*	7	$L : BA + ba :: BA - ba : SA - Sa$
Or	1	$L : ED :: DB : bD$

{ Which gives the following *Analogy.*

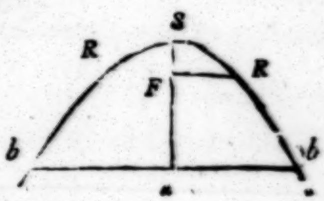
This

This peculiar Property of the Parabola was firſt publiſh'd *Anno* 1684, by one Mr. *Thomas Baker*, Rector of *Biſhop Nympton* in *Devonſhire*, in a Treatiſe entituled, *The Geometrical Key: Or, the Gate of Equations unlock'd*; wherein he hath ſhewed the Geometrical Conſtruction and Solution of all Cubic and Biquadratic Adfected Equations by one general Method, which he calls a *Central Rule*, deduced from this peculiar Property of the Parabola.

Sect. 3. *To find the* Focus *of any Parabola.*

THE *Focus* of every *Parabola* is that *Point* in its *Axis* thro' which the *Latus Rectum* doth paſs. (See *Definition* 3. *Sect.* 4. *page* 365.) Therefore its Diſtance from the *Vertex* of the *Parabola* may be eaſily found, either by the *Latus Rectum* itſelf, or by any other *Ordinate*, and its *Abſciſſa*.

Thus, ſuppoſe the *Point* at F to be the *Focus*, S the *Vertex*, the *Ordinate* $R\ F\ R = L$ the *Latus Rectum*, and $b\ a\ b$ any other *Ordinate*. Then will $S\ F = \frac{1}{4} L$.



Or $S\ F = \dfrac{\square\ b\ a}{4\ S\ a}$

Demonstration.

Firſt	1	$SF \times L = \square FR$, by Sect. 2. Page 381.
And	2	$FR = \frac{1}{2} L$; for the Ordinate $RFR = L$ as above.
2 ⊙ 2	3	$\square FR = \frac{1}{4} \square L = \frac{1}{2} L \times \frac{1}{2} L$
1, + 2	4	$SF \times L = \frac{1}{4} \square L$
4 ÷ L	5	$SF = \frac{1}{4} L$, as by Definition 4. Sect. 4. Page 365.
Again	6	$\dfrac{\square ba}{S\ a} = L$, by the 3d Step in Page 381.
Conſeq.	7	$\dfrac{\square ba}{4\ Sa} = \frac{1}{4} L$, &c. as above. Q. E. D.

Sect. 4. *To* Describe, *or draw a Parabola ſeveral Ways.*

NOTE, There are two or three Ways of drawing a Parabola inſtrumentally at one Motion; but becauſe thoſe Inſtruments or Machines are not only too perplex'd for a Learner to manage, but alſo a little ſubject to Error, I have therefore choſen to ſhew how that Figure may be (the beſt) drawn by a convenient Number of Points, *viz.* Ordinates found, either Numerically or Geometrically, according to the Data; which if the Work of the three laſt Sections be well conſidered, muſt needs be very eaſy.

1. If

1. If any *Ordinate* and its *Abscissa* are given, there may by them be found as many *Ordinates* as you please to assign or take *Points* in the *Parabola's Axis*; (by *Sect.* 4. *page* 380) and the *Curve* of the *Parabola* may be drawn by the extream Points of those *Ordinates*, as the *Ellipsis* was *page* 373.

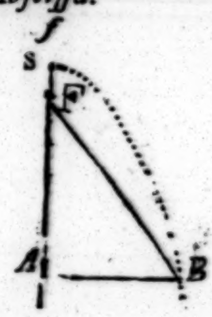
2. If the *Latus Rectum*, and either any *Ordinate*, or its *Abscissa*, are given, then any assign'd Number of *Ordinates* may by them be found (by *Sect.* 1. *page* 381.) either *Numerically* or *Geometrically*, &c.

3. If only the Distance of the *Focus* from the *Vertex* of the *Parabola* be given, any assign'd Number of *Ordinates* may be found by it. For $SF = \frac{1}{4}L$ the *Latus Rectum*, and $\frac{1}{2}L = FR$ as in the last *Section*; and it will be, as SF: is to $\square FR$: : so is any other *Abscissa*, viz. (Sa or SA, &c.) : to the *Square* of its *Semi-ordinate*, (viz. $\square ba$, or $\square BA$) according to the common Property of the *Parabola*.

Altho' any of these Ways of finding the *Ordinates* are easy enough, yet that Way which may be deduced from the following *Proposition* will be found more easy and ready in *Practice*.

PROPOSITION. *The Sum of any Abscissa and focal Distance from the Vertex, will be equal to the Distance from the Focus to the extream Point of the Ordinate, which cuts off that Abscissa.*

For Instance, suppose S to be the *Vertex* of any *Parabola*, the Point F to be its *Focus*, and AB any *Semi-ordinate* rightly apply'd to its *Axis* SA: Then I say, where-ever the *Point* A is taken in the *Axis*, it will be $SA + SF = FB$. Consequently, if $Sf = SF$, it will be $fA = FB$.



DEMONSTRATION

First	1	$SF=\frac{1}{4}L$ by the 7th Step, Sect. 3.
∴ *Ergo*	2	$fA=FA+\frac{1}{4}L$ by Construction above.
2 ⓖ^{2}	3	$\square fA=\square FA+FA\times L+\frac{1}{4}LL$
Again	4	$AS = FA+\frac{1}{4}L$ by the Supposition and Figure.
4 $\times L$	5	$SA\times L=FA\times L+\frac{1}{4}LL$, but $SA\times L=\square AB$
Ergo	6	$\square AB=FA\times L+\frac{1}{4}LL$
3 — 6	7	$\square fA-\square AB=\square FA$, conse. $\square fA=\square FA+\square AB$
But	8	$\square FA+\square AB=\square FB$, by Theor. 11.
Ergo	9	$\square fA=\square FB$
9 ɯ^2	10	$fA=FB$

Q. E. D.

This

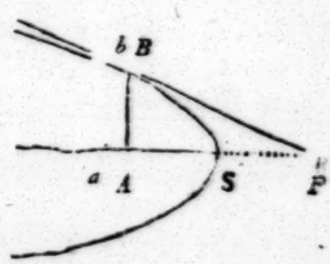
This *Proposition* being well understood, it will be very easily apply'd to *Practice*, supposing the *Focal Distance* given, or any other *Data* by which it may be found. Thus draw any *Right Line* to represent the *Parabola's Axis*, and from its *vertical Point*, as at *S*, set off the *Focal Distance* both upwards and downwards, *viz.* make $Sf = SF$, the Distance of the given *Focus* from the *Vertex*; as in the *Scheme*: Then by the *Proposition* it is evident, that, if never so many Lines be drawn *Ordinately* at *Right Angles* to the *Axis*, the true Distance between the Point *f* out of the *Parabola*, and any of those Lines (or *Ordinates*) being measur'd or set off from the *Focus F* to the same Line or Ordinate, it will assign the true Point in that Line through which the *Curve* must pass: that is, it will shew the true Limits or Length of that *Ordinate*; as at *B* in the last Scheme.

Proceeding on in the very same Manner from *Ordinate* to *Ordinate*, you may with great Expedition and Exactness find as many *Ordinates* (*or rather their Points only, like B*) as may be thought convenient, which, being all join'd together with an even Hand, will form the *Parabola* requir'd.

N. B. *The more* Ordinates (*or their* Points) *there are found, and the nearer they are to one another, the easier and exacter may the* Curve *of the* Parabola *be drawn. The same is to be observ'd when any other* Curve *is requir'd to be drawn by* Points.

Sect. 5. *To draw a* TANGENT *to any given* Point *in the* Curve *of a* Parabola.

Tangents are very easily drawn to the *Curve* of any *Parabola*; For, supposing *S* to be its *Vertex*, *B* the Point of *Contact* (*viz.* the Point where the *Tangent* must touch the *Curve*) and *P* the Point where the *Tangent* will intersect (*or meet with*) the *Parabola's Axis* produced: Then if from the Point of *Contact B* there be drawn the *Semi-ordinate B A* at Right Angles to the Axis *S A*, wheresoever the Point *A* falls in the Axis, it will be $SP = SA$.



DEMONSTRATION.

Draw the *Semi-ordinate* $b\ a$ (*as in the Figure*) then will the $\triangle BAP$ and $\triangle baP$ be alike. Let $y = AS$ the *Abscissa*, and

and $z = SP$; put $x = Aa$ the Distance between the two *Semi-ordinates*, which we suppose to be infinitely near each other, as in the *Ellipsis*, Page 377.

Then	1	$y+z : BA :: y+z+x : ba$, per *Theorem* 13.
1, Or	2	$y+z : y+z+x :: BA : ba$. See *Page* 192.
Again	3	$y : \square BA :: y+x : \square ba$, per *Theorem* Page 380.
3, Or	4	$y : y+x :: \square PA : \square ba$
2 in □'s	5	$\begin{cases} yy+2yz+zz+ : yy+2yz+2yz+zz+ \\ 2zx+xx :: \square BA : \square ba \end{cases}$
4, 5	6	$\begin{cases} y : y+x :: yy+2yz+zz : yy+2yz+ \\ 2yx+zz+2zx+xx \end{cases}$
6 ∵	7	$\begin{cases} yy+2yz+yx+zz+2zx+\frac{zzx}{y} = \\ yy+2yz+2yx+zz+2zx+xx. \end{cases}$
That is,	8	$\frac{zzx}{y} = yx+xx$, consequently $\frac{zz}{y}=y+x$
Suppose	9	$x = 0$ and rejected, as in the *Ellipsis*, Page 377.
Then	10	$\frac{zz}{y}=y$, consequently $zz=yy$
10 ɯ 2	11	$z=y$, that is, $SP = SA$

Q. E. D.

CHAP IV.

Concerning the chief Properties of the HYPERBOLA.

NOTE, any Part of the *Axis* of an *Hyperbola*, which is intercepted between its Vertex and any Ordinate (*viz.* any intercepted Diameter) is call'd an *Abscissa*; as in the *Parabola*.

Sect. I.

The Plane *of every* Hyperbola *is proportion'd by this general* Theorem.

THEOREM. As the Sum of the Transverse and any *Abscissa* multiply'd into that *Abscissa* : is to the Square of its Semi-ordinate :: so it is the Sum of the Transverse and any other *Abscissa* multiply'd into that *Abscissa* : to the Square of its Semi-ordinate.

That

That is, if TS be the Transverse Diameter,

And $\begin{cases} Sa, & SA \text{ Abscissæ.} \\ ba, & BA \text{ Semi-ordinates.} \end{cases}$

Then is $\begin{cases} TA = TS + Sa \\ TA = TS + SA \end{cases}$

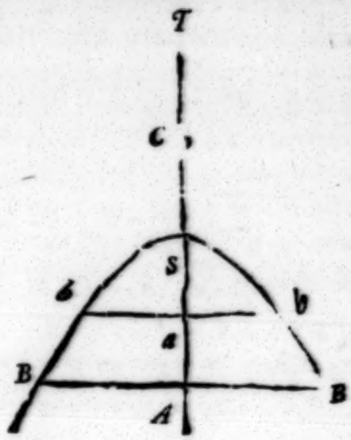
And it will be

$Ta \times Sa : \square\, ba :: TA \times SA : \square\, BA.$

That is,

$\overline{TS + Sa} \times Sa : \square\, ba :: \overline{TS + SA} \times SA : \square\, BA$

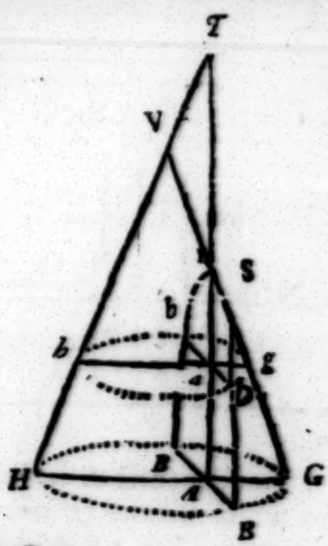
&c.



Demonstration.

Let the following Figure *HVG* represent a *Right Cone* cut into two Parts by the Right Line *SA*; then will the Plane of that Section be an *Hyperbola* (by *Sect*. 5. *Chap*. 1) in which let *S A* be its Axis, or intercepted Diameter, *b a b* and *B A B Ordinates* rightly apply'd (as before in the *Parabola*) and *T S* its Transverse Diameter. Again, if the Cone is suppos'd to be cut by *b g*, parallel to its Base *H G*, it will also be the Diameter of a Circle, &c. as in the *Ellipsis* and *Parabola*. Then will the $\triangle\, S\, g\, a$ and $\triangle\, S\, G\, A$ be alike; also the $\triangle\, T\, a\, b$ and $\triangle\, T\, A\, H$ will be alike; therefore it

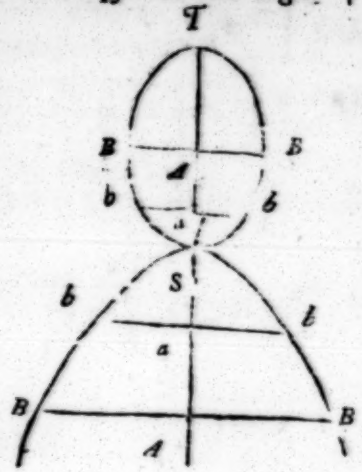
will be	1	$Sa : ag :: SA : AG$
And	2	$Ta : ab :: TA : AH$
1 ∴	3	$Sa \times AG = SA \times ag$
2 ∴	4	$Ta \times AH = TA \times ab$
3×4	5	$Sa \times Ta \times AG \times AH = SA \times TA \times ag \times ab$
But	6	$ag \times ab\ \square\, ab$
And	7	$AG \times AH = \square\, AB$ per *Lemma* Page 363
5, 6, 7	8	$Sa \times Ta \times \square\, AB = SA \times TA \times \square\, ab$ which give the following
8. *Anal.*	9	$Sa \times Ta : \square\, ab :: SA \times TA : \square\, AB$, &c.



Q. E. D.

These

These *Proportions* are the common Property of every *Hyperbola*, and do only differ from those of the *Ellipsis* in the *Signs* + and —; as plainly appears in the following *Proportions*. That is, if we suppose TS the Transverse Diameter common to both Sections (*viz.* both the *Ellipsis* and *Hyperbola*) as in the annexed Scheme: then in the *Ellipsis* it will be $\overline{TS - Sa} \times Sa : \square\, ab :: \overline{TS - SA} \times SA : \square\, AB$ as by *Sect.* 1, *Chap.* 2. and in the *Hyperbola* it is $TS + \overline{Sa} \times Sa : \square\, ab :: TS + SA \times SA : \square\, AB$, as above. Therefore all, that is farther requir'd in the *Hyperbola*, may (in a manner) be found as in the *Ellipsis*, due Regard being had to changing of the *Sign*.



Sect. 2. *To find the* LATUS RECTUM, *or* RIGHT PARAMETER, *of any Hyperbola.*

FROM the last Proportion take either of the Antecedents and its Consequent, *viz.* either $Ta \times Sa : \square\, ab$. Or $TA \times SA : \square\, AB$, to them bring in the Transverse TS for a third Term, and by those three find a fourth Proportional (as in the Ellipsis) and that will be the *Latus Rectum.*

Thus	1	$\left\{ Ta + Sa : \square\, ab :: TS : \dfrac{\square\, ab + TS}{Ta \times Sa} \right.$ — = the *Latus Rectum*, which call L (*as in the* Parabola.)
Then	2	$TS : L :: Ta \times Sa : \square\, ab$.
But	3	$Ta \times Sa : \square\, ab :: TA \times SA : \square\, AB$, therefore
2, 3	4	$TS : L :: TA \times SA : \square\, AB$, &c.

Consequently L is the true *Latus Rectum*, or right Parameter, in which all the *Ordinates* may be found, according to its Definition in *Chap.* 1. And because $TS + Sa = Ta$, let it be $TS + Sa$ instead of Ta, then it will be $\dfrac{\square\, ab \times TS}{TS \times Sa + \square\, Sa} = L$ and in the *Ellipsis* it would be $\dfrac{\square\, ab \times TS}{TS \times Sa - \square\, Sa} = LR = L.$

Sect,

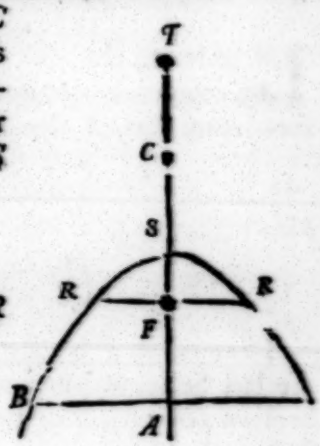
Sect. 3. *To find the* Focus *of an Hyperbola.*

THE *Focus* being that Point in the *Hyperbola*'s Axis through which the *Latus Rectum* must pass (*as in the* Ellipsis *and* Parabola) it may be found by this *Theorem*.

Theorem. *To the Rectangle made of half the Transverse into half the* Latus Rectum, *add the Square of half the Transverse; the Square Root of that Sum will be the Distance of the* Focus *from the Center of the* Hyperbola.

Demonstration.

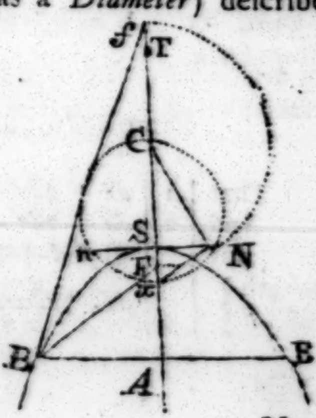
Suppose the Point at *F*, in the annex'd Scheme, to be the *Focus* sought; then will $FR = \frac{1}{2} L$. Let $TC = CS$ be half the Transverse; then is the Point *C* call'd the *Center* of the *Hyperbola* (for a Reason that shall be hereafter shew'd.) Again; let $CS = d$. and $SF = a$.



Then	1	$2d : L :: \overline{2d+a} \times a : \frac{1}{4} LL$
That is,	2	$TS : L :: \overline{TS+SF} \times FS : \square FR$
1 ∵	3	$\frac{1}{2} dL = 2da + aa$
3 + dd	4	$dd + \frac{1}{2} dL = dd + 2da + aa$
4 ɯ^2	5	$\sqrt{dd + \frac{1}{2} dL} = d + a = FC$
Or 5, — d	6	$\sqrt{dd + \frac{1}{2} dL} - d = a = SF$

In the *Ellipsis* it is, $2d : L :: \overline{2d-a} \times a : {}^{1} LL$. that is, $\frac{1}{2} d L = 2da - aa$, &c.

The Geometrical Affection of the last Theorem is very easily perform'd, thus: make $Sx = \frac{1}{2} L$, *viz.* half the *Latus Rectum*; and let $CS = d$, as above. Upon Cx (as a *Diameter*) describe a Circle, and at *S* the *Vertex* of the *Hyperbola* draw the Right Line $n\,S\,N$ at Right Angles to Cx; then join the Points CN with a Right Line, and it will be $CN = d + a = FC$.



For	1	$CS : SN :: SN : Sx$. per *Fig.*
That is	2	$d : SN :: SN : \frac{1}{2} L$.
2 ∵	3	$\frac{1}{2} d L = \square SN$
But	4	$ad + \square SN = \square CN$
3, 4	5	$dd + {}^{1} dL = \square CN$
5 ɯ^2	6	$\sqrt{ad + \frac{1}{2} dL} = CN = d + a$ &c.

Now

Now here is not only found the Distance of the *Hyperbola's Focus*, either from its Center C. or Vertex S, but here is also found that Right Line usually call'd its Conjugate Diameter, *viz.* the Line $n\ S\ N$, which bears the same Proportion to the Transverse and *Latus Rectum* of the *Hyperbola*, as the Conjugate Diameter of the *Ellipsis* doth to its Transverse and *Latus Rectum*. For in the *Ellipsis* $TS : Nn :: Nn : LR$. per *Sect.* 2, *Page* 369. Consequently $\frac{1}{2}\ TS : \frac{1}{2}\ Nn :: \frac{1}{2}\ Nn : \frac{1}{2}\ LR$. But $\frac{1}{2}\ TS = d$, $\frac{1}{2}\ Nn = SN$, and $\frac{1}{2}\ LR = \frac{1}{2}\ L$. Therefore $d : SN :: SN : \frac{1}{2}\ L$. As at the 2d Step above.

What Use the aforesaid Line $n\ SN$ is of, in Relation to the *Hyperbola*, will appear farther on.

Sect. 4. *To describe an* Hyperbola *in Plano.*

IN order to the easy Describing of an *Hyperbola* in *Plano*, it will be convenient to premise the following *Proposition*, which differs from that of the *Ellipsis* in *Sect.* 3, *Chap.* 2. only in the *Signs*.

PROPOSITION. *If from the* Foci *of an* Hyperbola *there be drawn two Right Lines, so as to meet each other in any Point of the* Hyperbola's *Curve, the Difference of those Lines* (in the *Ellipsis* it is their Sum) *will be equal to the Transverse Diameter.*

That is, if F be the *Focus*, and it be made $C\ f = C\ F$ (as in the last Scheme) then the Point f is said to be a *Focus* out of the Section (or rather of the opposite Section) and it will be $f\ B - F\ B = T\ S$.

DEMONSTRATION.

Suppose $f\ C$, or $C\ F$, $= z$, and $S\ A = x$, let $C\ S$, or $T\ C = d$, as before; then will $f\ A = d + x + z$, and $F\ A = d + x - z$. Again, let $F\ B = b$, and $f\ B = b$, then $2d = b - b$, by the *Proposition*.

From these *substituted Letters*, it follows,

That	1	$dd + 2dx + 2dz + xx + 2zx + zz = \square f\ A$
And	2	$dd + 2dx - 2dz + xx - 2zx + zz = \square F\ A$
But		$\square f\ A + \square\ AB = \square f\ B$, and $\square\ F\ A + \square\ A\ B = \square FB$
Per 4th of last	3	$dd + \frac{1}{2}\ dL = da + 2da + aa = \square\ FC = zz$.

3 — dd

3 − dd	4	$zz - dd = \frac{1}{2}d\,L$
4 ÷ $\frac{1}{2}d$	5	$\frac{zz-dd}{\frac{1}{2}d} = L$
Again	6	$2d : L :: \overline{2d + x} \times x : \square AB$, By common Properties.
5, 6	7	$2d : \frac{zz-dd}{\frac{1}{2}d} :: 2dx + xx : \square AB$
7 ⁚	8	$\frac{2dzzx + zzxx - 2dddx - ddxx}{dd} = \square AB$
1 + 8	9	$\left\{ dd + 2dx + 2dz + xx + 2zx + zz + \frac{2dzzx + zzxx - 2d^3x - ddxx}{dd} \right. = \square FA + \square AB = bb$
2 + 8	10	$\left\{ dd + 2dx - 2dz + xx - 2zx + zz + \frac{2dzzx + zzxx - 2d^3x - ddxx}{dd} \right. = \square FA + \square AB = bb$
9 + d	11	$d^4 + 2d^3z + 2ddzx + ddzz + 2dzzx + zzxx = ddbb$
10 × dd	12	$d^4 - 2d^3z - 2ddzx + ddzz + 2dzzx + zzxx = ddbb$
11 ω^2	13	$dd + dz + zx = db$
12 ω^2	14	$dd + dz - zx = db$
13 ÷ d	15	$d + z + \frac{zx}{d} = b$
14 ÷ d	16	$d - z - \frac{zx}{d} = b$
16, or	17	$z + \frac{zx}{d} - d = b$
15 − 17	18	$2d = b - b$

Altho' the Æquation at the 16*th* Step *be in itself impossible, because* z *is greater than* d (*by the* 4*th* Step) *yet from thence it will be easy to conclude, that the Difference between* d *and* $z + \frac{zx}{d}$ *will give the true Value of* b, *as in the* 17*th* Step.

But becauſe I would leave no Room for the Learner to doubt about changing the *Æquation*, $d - z - \frac{zx}{d} = b$ into that of $x + \frac{zx}{d} - d = b$. it may be convenient to illuſtrate the whole Proceſs in Numbers, whereby (I preſume) it will plainly appear that $b - b = TS$.

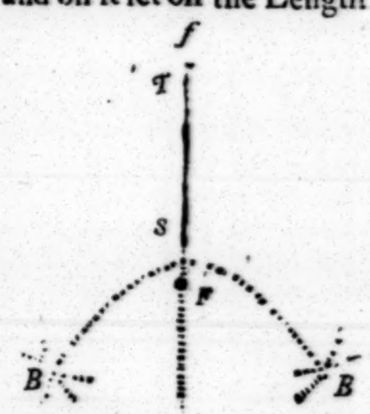
In order to that, let the Tranſverſe $TS = 2d = 12$, then $d = 6$ ſuppoſe the Abſciſſa $SA = x = 4$, and the Semi-ordinate $AB = 3$

Firſt	1	$TS + SA \times SA : \square AB :: TS : L$, *per* Sect. 2.
1, *viz.*	2	$12 + 4 \times 4 = 64 : 9 :: 12 : 1{,}6875 = L$
Again	3	$\sqrt{dd + \frac{1}{4}dL} = d + a = CF$. *per* Sect. 3.
2, *viz.*	4	$\sqrt{36 + 5{,}0625} = 6{,}408 = CF = z$
Then	5	$a + x + z = 6 + 4 + 6{,}408 = 16{,}408 = fA$
And	6	$d + x - z = 6 + 4 - 6{,}408 = 3{,}592 = FA$

5 ⊙^{2}	7	$269,2224 = 2fA$
6 ⊙^{2}	8	$12,9024 = \square FA$
But	9	$9 = \square AB$, for $AB = 3$ By Supposition.
$7 + 9$	10	$278,2224 = \square fA + \square AB = \square fB$
$8 + 9$	11	$21,9024 = \square FA + \square AB = \square FB$.
10 ɯ^{2}	12	$16,68 = fB$
11 ɯ^{2}	13	$4,68 = FB$
$12 - 13$	14	$12,00 = fB - FB = TS$. Which was to be prov'd.

If this *Proposition* be truly underſtood, it muſt needs be eaſy to conceive how to deſcribe the Curve of any *Hyperbola* very readily by *Points*, when the *Tranſverſe Diameter* and the *Focus* are given (or any other *Data* by which they may be found, as in the precedent Rules) thus:

Draw any ſtreight Line at Pleaſure, and on it ſet off the Length of the given *Tranſverſe*, *T S*, and from its extreme Points or Limits, *viz.* *T S*, ſet off *T f*, = *S F*, the Diſtance of the given *Focus*, (viz. *the Point f without, and F within the Section, as before*): that done, upon the *Point f* (*as a Center*). with any aſſum'd *Radius* greater than *T S*, deſcribe an Arch of a Circle; then from that *Radius* take the Tranſverſe *T S*, making their Difference a ſecond *Radius* with which, upon the Point *F*, within the Section, deſcribe another Arch to cut or croſs the firſt Arch, as at *B*; then will that Point *B* be in the Curve of the *Hyperbola*, by the laſt Propoſition. And therefore it is plain, that, proceeding on in this Manner, you may find as many Points (like *B*) as may be thought convenient (the more there are, and the nearer they are together, the better) which being all join'd together with an even Hand (as in the Parabola) will form the *Hyperbola* requir'd.

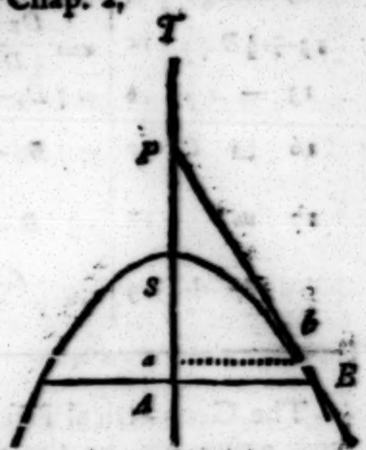


There are ſeveral other Ways of delineating an *Hyperbola* in *Plano*: One Way is, by finding a competent Number of *Ordinates*, as by Section 1, &c. but I think none ſo eaſy and expediditious as this mechanical Way: I ſhall therefore, for Brevity's Sake, paſs over the reſt, and leave them to the Learner's Practice, as being eaſily deduced from what hath been already ſaid.

Sect.

Sect. 5. *To draw a* TANGENT *to any given Point in the Curve of an* HYPERBOLA.

The drawing of a *Tangent*, that will touch any given Point in the Curve of an *Hyperbola*, may be easily performed by Help of a *Theorem*; as in the *Ellipsis*, Sect. 6, Chap. 2.



Let $\left\{\begin{array}{l} D = TS \text{ the } \textit{Transverse Diameter.} \\ L = \text{the } \textit{Latus Rectum.} \\ y = SA \text{ the } \textit{Abscissa.} \end{array}\right.$

And $z = AP$ {the Distance between the *Ordinate* and that Point in the *Transverse* cut by the *Tangent*.

Then, if y be given, z may be found by this *Theorem*, $\left\{ \dfrac{Dy+yy}{\frac{1}{2}D+y} = z \right.$ [which differs from that in the *Ellipsis* only in Signs. *Vide page* 377.]

Or, if z be given, then y may be found by this *Theorem*;

THEOREM. $\sqrt{\dfrac{DD+zz}{4}} : +\frac{1}{2}z - \frac{1}{2}D = y.$

DEMONSTRATION.

Draw the *Semi-ordinate* $b\,a$, as in the Figure, and put $x = Aa$ $\left\{\right.$ an infinite small Space between the two *Semi-ordinates*; as before in the *Ellipsis*, &c.

Then	1	$D : L :: Dy+yy : \square AB$
That is,	2	$TS : L :: \overline{TS+SA} \times SA : \square AB$
1 ∵	3	$\dfrac{Dy\,L+yy\,L}{D} = \square AB$
Again	4	$D : L :: Dy+yy-2yx-Dx+xx : \square ab$
That is,	5	$TS : L :: TS+Sa \times Sa : a\square b$
4 ∵	6	$\dfrac{DyL+yyL-2yxL-DxL+xxL}{D} = \square ab$
Per Figure	7	$z : AB :: z-x : ab$, viz. $PA : AB :: Pa : ab$
7 in □'s	8	$zz : \square AB :: zz-2zx+xx : \square ab$
Suppose	9	$x=0$ and every where rejected (as in the *Ellipsis*.)
Then 3, 9	10	$zz : \dfrac{DyL+yyL}{D} :: zz-2z : \square ab$
10 ∵	11	$\dfrac{Dy\,L\,zz+yy\,L\,zz-2DyLz-2yyLz}{Dzz} = \square ab.$

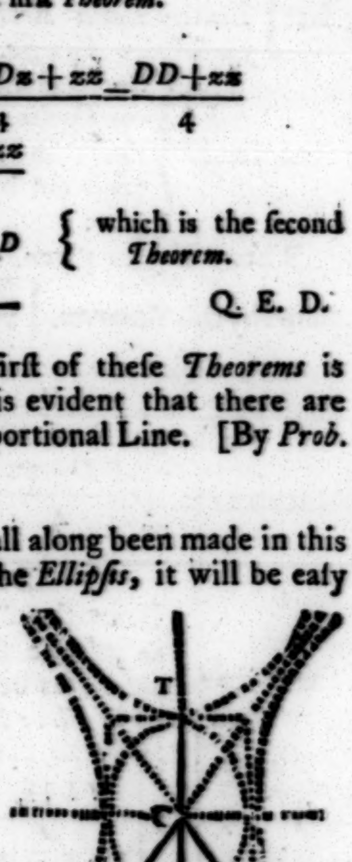
6, 11	12	$\left\{\frac{Dy\,L+yy\,L-2y\,L-DL}{D}=\frac{Dy\,Lzz+yy\,Lzz-2\,Dy\,Lz-2yy\,Lz}{Dzz}\right.$
12 reduced	13	$\frac{1}{2}Dz+zy=Dy+yy$
13 *Analogy*	14	$\frac{1}{2}D+y : y :: D+y : z$, viz. $CA : SA :: TA : AP$
$13 \div \frac{1}{2}D+y$	15	$z=\frac{Dy+yy}{\frac{1}{2}D+y}$ which is the first *Theorem.*
$13 - zy$	16	$yy+Dy-zy=\frac{1}{2}Dz$
16 $\square$ C	17	$yy+Dy-yz+\frac{DD-2\,Dz+zz}{4}=\frac{DD+zz}{4}$
17 ww^{2}	18	$y+\frac{1}{2}D-\frac{1}{2}z=\sqrt{\frac{DD+zz}{4}}$
18 $\pm$	19	$y=\sqrt{\frac{D+zz}{4}}:+\frac{1}{2}z-\frac{1}{2}D$ { which is the second *Theorem.*

Q. E. D.

The Geometrical Effection of the first of these *Theorems* is very easy; for, by the 14th Step, it is evident that there are three Lines given to find a fourth proportional Line. [By *Prob.* 3, *page* 308.]

Scholium.

From the Comparisons, which have all along been made in this Chapter, between the *Hyperbola* and the *Ellipsis*, it will be easy (even for a Learner) to perceive the Coherence that is in (or between) those two Figures, but, for the better understanding of what is meant by the *Center* and *Asymptotes* of an *Hyperbola*, consider the annexed *Scheme*, wherein it is evident (even by Inspection) that the opposite *Hyperbola*'s will always be alike, because they will always have the same Transverse Diameter common to both, &c. (see *Sect.* 1, of this *Chap.*). Also, that the middle Point, or common Center of the *Ellipsis*, is the common Center to all the four *conjugal Hyperbola*'s.



And the two Diagonals of the Right-angled *Parallelogram*, which circumscribes the *Ellipsis* (or is inscrib'd to the four *Hyperbola*'s) being continued, will be such *Asymptotes* to those *Hyperbola*'s as are defined *Chap.* 1, *Sect.* 5, *Defin.* 4.

Sect.

Sect. 6. *To draw the* Asymptotes *of any* Hyperpola, &c.

Having found the *Latus Rectum* (by Sect. 2.) and the Conjugate Diameter $n\,S\,N$ in its true *Position*, by Sect. 3. Then thro' the Center C of the *Hyperbola*, and the extream Points $n\,N$ of its Conjugate Diameter, draw two *Right Lines*, as $C\,N$ and $C\,n$, infinitely continued (as in the following Figure) and they will be the *Asymptotes* required. That is, they are two such *Right Lines* as, being infinitely extended, will continually incline to the Sides of the *Hyperbola*, but never touch them.

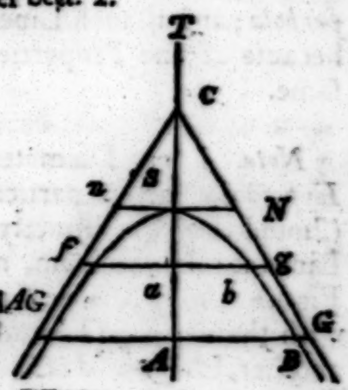
Demonstration.

Suppose the *Semi-ordinates* $a\,b$ and AB to be rightly apply'd to the *Axis* $T\,A$; and produced both Ways to the *Asymptotes*, as at $f\,g$ and $F\,G$; then will the $\triangle\, C\,S\,N$, $\triangle\, C\,a\,g$, and $\triangle\, C\,A\,G$ be alike.

Let $d = C\,S = T\,C$. And $L =$ the *Latus Rectum*; as before.

Put $\left\{\begin{array}{l} e = S\,a \\ y = SA \end{array}\right\}$ the *Abscissæ*. Then $\left\{\begin{array}{l} d + e = C\,a. \\ d + y = CA. \end{array}\right.$

The	1	$d : SN :: d+e : ag$. viz. $CS : SN :: Ca : ag$
1 in □'s	2	$dd : \square SN :: dd+2de+ee : \square ag$.
But	3	$\frac{1}{2}dL = \square SN$. per Sect. 3.
2 3 ∵	4	$\dfrac{dd\,L + 2de\,L + ee\,L}{2d} = \square ag$
Again	5	$2d : L :: 2de+ee : \square\, ab$, per Sect. 2.
5 ∵	6	$\dfrac{2deL+eeL}{2d} = \square\, ab$
4 — 6	7	$\dfrac{dL}{2} = \square ag - \square ab$
But {	8	$ag + ab = bf$ } per *Fig.*
	9	$ag - ab = bg$ }
8 × 9	10	$\square ag - \square ab = bf \times bg$
7, 10	11	$bf \times bg = \frac{1}{2}dL$
Again	12	$dd : \square SN :: dd+2dy+yy : \square AG$
That is,		$\square CS : \square SN :: \square CA : \square AG$
3, 12 ∵	13	$\dfrac{adL + 2dyL + yyL}{2d} = \square AG$
But	14	$2d : L :: 2dy+yy : \square AB$, per Sect. 2.
14 ∵	15	$\dfrac{2dy\,L + yy\,L}{2d} = \square\, AB$



13 — 15	16	$\frac{dL}{2} = \square AG - \square AB$
Also	17	$AG + AB = BF$ per Fig.
	18	$AG - AB = BG$ per Fig.
17 × 18	19	$\square AG - \square AB = BF \times BG$
16 19	20	$BF \times BG = \frac{1}{2} dL$
11, & 20 ÷	21	$bg = \frac{\frac{1}{2} dL}{bf}$ And $BG = \frac{\frac{1}{2} dL}{BF}$

From the last Step it is evident, that the *Asymptotes* are nearer the *Hyperbola* at G than at *g*, and consequently will continually approach to its Curve: For $BF) \frac{1}{2} dL (= BG$ is less than $bf) \frac{1}{2} dL (= bg$, because the *Divisor* BF is greater than the *Divisor* bf; and it must needs be so where-ever the *Ordinates* are produced to the *Asymptotes*, from the Nature of the *Triangles*.

Again; from the 7th and 16th Steps it is evident, that the *Asymptotes* can never really meet and be co-incident with the Curve of the *Hyperbola*, altho' both were infinitely extended, because $\frac{1}{2} dL$ will always be the Difference between the Square of any *Semi-ordinate*, and the Square of that *Semi-ordinate* when it is produced to the *Asymptote*.

Consectary.

From hence it follows, that every *Right Line* which passes thro' the *Center* and falls within the *Asymptotes*, will cut the *Hyperbola*; and all such Lines are call'd *Diameters* (as in the *Ellipsis*) because of the Properties of the *Hyperbola* and *Ellipsis* are the same.

Note. Every Diameter, both in the *Ellipsis*, *Parabola*, and *Hyperbola*, hath its particular *Latus Rectum* and *Ordinates*; which (should they be distinctly handled, and the Effection of all such Lines as relate to them, as also the Nature and Properties of such Figure as may be inscribed and circumscribed to all the Sections, with the various Habitudes or Proportions of one *Hyperbola* to another, &c.) would afford Matter sufficient to fill a large Volume. But thus much may suffice by way of *Introduction*; I shall therefore desist pursuing them any farther, being fully satisfied, that, if what I have already done be well understood, the rest must needs be very easy to any one that intends to proceed farther on that Subject.

AN

AN INTRODUCTION TO THE MATHEMATICKS.

PART V.

THE Method of finding out any particular Quantity (*viz.* either any LINE, SUPERFICIES, or SOLID) by a regular Progression, or Series of Quantities continually approaching to it, which, being infinitely continued, would then become perfectly equal to it; is what is commonly call'd *Arithmetick of Infinites*; which I shall briefly deliver in the following *Lemma's*, and apply them to Practice in finding the superficial and solid Contents of Geometrical Figures farther on.

LEMMA I.

If any Series of equal Numbers (representing Lines or other Quantities (as, 1. 1. 1. 1. &c. or 2. 2. 2. 2. &c. or 3. 3. 3. 3. &c. if one of the Terms be multiply'd into the Number of Terms, the Product will be the Sum of all the Terms in the Series.

This is so very plain, and easy to be understood, that it needs no *Example*.

LEMMA II.

If a Series of Numbers in Arithmetick Progression begin with a Cypher, and the common Difference be 1; as, 0. 1. 2. 3. 4. &c. (representing a Series of Lines or Roots beginning with a Point) if the last Term be multiply'd into the Number of Terms, the Product will be double the Sum of all the Series.

That is, putting $L =$ the last Term, $N =$ the Number of Terms, and $S =$ the Sum of all the Series: Then

Then will $NL = 2S$. Consequently, $\frac{1}{2}NL = S$. *viz.* one Half of so many Times the greatest Term as there are Numbers of Terms in the Series.

Thus $\frac{0+1+2+3+4}{4+4+4+4+4} = \frac{10 = \text{the Sum of the Series} = \frac{1}{2}NL.}{20 = NL.}$

And this will always be so, how many Terms soever there are, by *Consect.* 1, *page* 185.

LEMMA III.

If a Series of Squares whose Sides or Roots are, in Arithmetick Progression, beginning with a Cypher, &c. (as in the last *Lemma)* be infinitely continued; the last Term being multiply'd into the Numbers of Terms will be Triple to the Sum of all the Series, *viz.* $NLL = 3S$, or $\frac{1}{3}NLL = S$.

That is, the Sum of such a Series will be one Third of the last or greatest Term, so many Times repeated as is the Number of Terms in the Series.

Instances in Square Numbers.

1. $\left\{\frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}\right.$

2. $\left\{\frac{0+1+4+9}{9+9+9+9} = \frac{14}{30} = \frac{7}{18} = \frac{1}{3} + \frac{1}{18}\right.$

3. $\left\{\frac{0+1+4+9+16}{16+16+16+16} = \frac{30}{80} = \frac{3}{8} = \frac{9}{24} = \frac{1}{3} + \frac{1}{24}\right.$ &c.

From these Instances it is evident, that as the Number of Terms in the Series does increase, the Fraction or Excess above $\frac{1}{3}$ does decrease, the said Excess always being $\frac{1}{6N-6}$; which, if we suppose the Series to be infinitely continued, will then become infinitely small, *viz.* in Effect nothing at all. Consequently, $\frac{1}{3}NLL$ may be taken for the true or perfect Sum of such an infinite Series of Squares.

LEMMA IV.

If a Series of Cubes whose Roots are in Arithmetick Progression, beginning with a Cypher, &c. (as above) be infinitely continu'd, the Sum of all the Series will be $\frac{1}{4}NLLL = S$.

That is, one Fourth of the last or greatest Term so many Times repeated as is the Number of Terms.

Instances

Instances in Cube Numbers.

If 0. 1. 2. 3. *&c.* be the *Roots* of the *Cubes*.

$$\text{Then 1.}\left\{\frac{0+1+8+27}{27+27+27+27}=\frac{36}{108}=\frac{4}{12}=\frac{1}{4}+\frac{1}{12}\right.$$

$$\text{2.}\left\{\frac{0+1+8+27+64}{64+64+64+64+64}=\frac{100}{320}=\frac{10}{32}=\frac{5}{16}=\frac{1}{4}+\frac{1}{16}\right.$$

$$\text{3.}\left\{\frac{0+1+8+27+64+125}{125+125+125+125+125+125}=\frac{225}{750}=\frac{45}{150}=\frac{3}{10}=\frac{6}{20}=\frac{1}{4}+\frac{1}{20}\right.$$

From these *Examples* it plainly appears, that, as the Number of Terms in the Series increases, the Fraction or Excess above $\frac{1}{4}$ decreases, the Excess being always $\frac{1}{4N-4}$; which, if we suppose the Series to be infinitely continued, will become infinitely small, or rather nothing; as in the last *Lemma*. Consequently, $\frac{1}{4}$ *NLLL* may be taken for the true and perfect Sum of all the Terms in such an infinite Series of Cubes.

LEMMA V.

If a Series of Biquadrates, whose Roots are in Arithmetick Progression, beginning with a Cypher, *&c.* (as before) be infinitely continued, the Sum of all the Terms in such a Series will be $\frac{1}{5}NL^4$.

The Truth of this may be manifested by the like Process as in the foregoing *Lemma*'s, and so on for higher Powers. But if any one desires a farther Demonstration of these Series, he may (I presume) meet with ample Satisfaction in Dr. *Wallis*'s Hist. of *Algeb.* Ch. 78 & 79, wherein the *Doctor* concludes with these Words:

" Thus having shewed, that in the Progression of Laterals (or " Arithmetical Proportionals) beginning at 0. the Sum of 2. 3. 4. " 5. 6 Terms, is always equal to half of so many times the great- " est; and there being no Pretence of Reason why we should " then doubt it in a Progression of 7. 8. 9. 10. *&c.* we conclude " it so to be, tho' such Number of Terms be supposed infinite.

" Again; in a Progression of their Squares having shew'd, that " in 2. 3. 4. 5. 6 Terms the Aggregate is always more than one " Third of so many times the greatest, and the Excess always such

" aliquot

" aliquot Part of the greatest, as is denominated by six times the " Number of Terms wanting 1. (As, if the Terms be 2, " it is $\frac{1}{3}+\frac{1}{6}$; if three, it is $\frac{1}{3}+\frac{1}{12}$; if 4, it is $\frac{1}{3}+\frac{1}{18}$; if 5, it " is $\frac{1}{3}+\frac{1}{24}$ of so many times the greatest Term, and so onward) " we may well conclude (there being no Pretence of Reason " why to doubt it in the rest) that it will be so, how many soever " be such Number of Terms. And because such Excess, as the " Number of Terms do encrease, will become infinitely small (or " less than any assignable) we conclude (from the Method of Ex- " haustions) that, if the Number of Terms be supposed infinite, " such Excess must be supposed to vanish, and the Aggregate of " such infinite Progressions suppos'd equal to $\frac{1}{3}$ of so many times " the greatest.

" In like Manner having proved that such Progression of " Cubes doth (as the Number of Terms encrease) approach in- " finitely near to $\frac{1}{4}$ of so many Times the greatest, and of Biqua- " drates to $\frac{1}{5}$, and so of Sursolids to $\frac{1}{6}$ of so many Times the " greatest, and so onwards as we please to try; and there being " no Pretence of Reason, why to doubt it as the rest, we may " take it as a sufficient Discovery, that (universally) the Aggre- " gate of such infinite Progression is equal (or doth approach in- " finitely near) to such a Part of so many Times the greatest, as " is denominated by the Exponent (or Number of Dimensions) " of such Power (as is that according to which the Progression " is made) encreased by 1. namely, of Laterals $\frac{1}{2}$; of Squares $\frac{1}{3}$; " of Cubes $\frac{1}{4}$; of Biquadrates $\frac{1}{5}$; (of so many Times the great- " est) and so onwards infinitely."

This Discourse of the *Doctor*'s I thought convenient to insert, supposing it may give some Satisfaction to the Learner, to hear so Great a Man as Dr. *Wallis*'s Argument about the Truth of these Series, which I have briefly delivered in the 'foregoing *Lemma*'s.

LEMMA VI.

If any two Series or Ranks of Proportionals have the same Number of Terms (whether Finite or Infinite) it will always be { As the first Term of one Series : is to the first Term of the other Series : : so is the Sum of all the Terms in the one Series : to the Sum of all the Terms in the other Series.

(12. *e.* 5.)

As

As in these Numbers, Or these Numbers,

1	3	4	5
2	6	12	15
3	9	36	45
4	12	108	135
5	15	324	405
6	18	972	1215

That is, 1 : 3 : : 21 : 63 And 4 : 5 : : 1456 : 1820 &c.

The Application of these *Lemmas* to Geometrical Quantities, *viz.* to Lines, Superficies, and Solids, wholly depends upon granting the following *Hypotheses.*

The Hypotheses.

1. That every Line is supposed to consist (or be composed) of an infinite Series of equidistant Points.

2. A Surface (*viz.* the Area of any Figure) to consist of an infinite Series of Lines, either streight or crooked, according as the Figure requires.

3. A Solid to consist of an infinite Series of Planes, or Superficies, according as its Figure requires.

Not that we suppose Lines, which have really no Breadth, can fill a Space or Superficies; or that Planes, which have not any Thickness, can constitute a Solid: But by what we here call Lines are to be understood small Parallelograms (or other Superficies) infinitely narrow, yet so as that their Breadths, being all taken and put together, must be equal to the Figure they are supposed to fill up. And those Planes or Superficies, which are here said to constitute a Solid, are to be understood infinitely thin; yet so as that their Depths or Thicknesses (which are hereafter also called Lines) being all taken together, must be equal to the Height of the proposed Solid. Now, in order to render this Hypothesis as easy for a Learner to understand as I can, I shall here propose a very plain and familiar Example; viz. Let us suppose any Book to be composed (or made up) of 100, 200, 300 (more or less) Leaves of fine Paper such a Book, being close put together, will have Length, Breadth, and Depth or Thickness, and therefore may (not improperly) be called a Solid; and each of its Edges (being evenly cut) will be a Superficies composed of a Series of small Parallelograms, every one of their Breadths being only the Edge of a single Leaf of Paper; and if we conceive the Thickness of every one of those Leaves to be divided into 10, or 100, or 1000, &c. they will then become such a Series of infinitely small Lines as are (by the Hypothesis) said to compose or

 fill

fill up a Superficies. And all the Superficies of thofe infinitely thin or divided Leaves of Paper will become fuch a Series of Planes, or Superficies, as are faid to conftitute a Solid, *viz.* fuch a Solid as the Bignefs and Figure of that Book.

Now according to this Idea of Lines, Superficies, and Solids, one may, without the leaft Prejudice to any *Demonftration*, admit of the following Definitions and Theorems.

Definitions.

I. The Areas of Squares, and all other Parallelograms, are compofed or filled up with an infinite Series of equal Right Lines.

II. The Area of every plain Triangle is compofed of an infinite Series of Right Lines parallel to its Bafe, and equally decreafing until they terminate in a Point at the vertical Angle.

III. The Area of a *Circle* may be compofed either of an infinite Series of concentric or parallel Circles, or of an infinite Series of Chord Lines parallel to its Diameter, or of an innumerable Multitude of Sectors.

IV. The Area of an *Ellipfis* may be compofed either of an infinite Series of Ordinates rightly applied, or of an infinite Series of Right Lines parallel to its Tranfverfe Diameter.

V. The Areas of the *Parabola* and *Hyperbola* are compofed of an infinite Series of Ordinates; or may alfo be compofed of Right Lines parallel to its Axis, &c.

VI. A *Prifm* is a folid Body contained or included within feveral equal Parallelograms, having its Bafe or Ends equal and alike; and it is generally named according to the Figure of its Bafe: That is,

VII. A *Cube* (or Solid like a Dye) is a Prifm bounded or included with fix equal fquare Planes.

VIII. A Parallelopipedon is a Prifm that hath its Sides bounded or included within four equal Parallelograms and two fquare Bafes or Ends.

IX. A Cylinder (or Solid, like a Rolling-ftone in a Garden) is only a round Prifm, having its Bafes or Ends a perfect Circle.

X. The Solidity of every Prifm is compofed of an infinite Series of equal Planes, parallel and alike to that of its Bafe.

XI. A

XI. A Pyramid is a Solid bounded or included within ſeveral plain Triangles ſet upon any polygonous Baſe, having their vertical Angles all meeting together in a Point, called the Vertex, and takes its Name from the Figure of its Baſe, viz. if it has a ſquare Baſe, 'tis called a ſquare Pyramid; if a triangular Baſe, 'tis called a triangular Pyramid. &c.

XII. A Cone is only a round Pyramid, which hath been already defined in Page 361, &c.

XIII. The Solidity of every Pyramid is compoſed or conſtituted of an infinite Series of Planes, parallel and alike to that of its Baſe, equally decreaſing until they terminate in a Point at the Vertex.

XIV. A Sphere or Globe, (viz. a Ball) is a Solid bounded or included within one regular Superficies, being formed or generated by the Rotation of a Semi-circle about its Diameter (called the Axis of a Sphere) and its Solidity is compoſed or conſtituted of an infinite Series of concentric Circles, whoſe Diameters are the Chords of that Circle by which it was formed.

XV. A Spheroid (or Egg-like Figure) is a Solid bounded with one regular Superficies, formed by the Rotation of a Semi-ellipſis about its tranſverſe Diameter (called the Axis of the Spheroid) and its Solidity is conſtituted of an infinite Series of concentric Circles, whoſe Diameters are the Ordinates of that Ellipſis by which it was formed.

XVI. There is another Sort of Solid called an *Oblate Spheroid*, being formed by the Rotation of an Ellipſis about its Conjugate Diameter, and it is like a flat Turnip.

XVII. If a Semi-parabola be turned about its Axis, it will form a Solid called a parabolic Conoid, being compoſed or conſtituted of an infinite Series of Circles, whoſe Diameters are the Ordinates of a Parabola.

XVIII. If a Parabola be turned about its Baſe, or greateſt Ordinate, it will form a Solid called a Pyramidoid, but moſt commonly a parabolic Spindle which will be conſtituted of an infinite Series of Circles whoſe Diameters are Right Lines parallel to the Parabola's Axis.

XIX. If an Hyperbola be turned about its Axis, it will form a Solid called an Hyperbolic Conoid, being conſtituted of an infinite Series of Circles whoſe Diameters are the Ordinates of the Hyperbola.

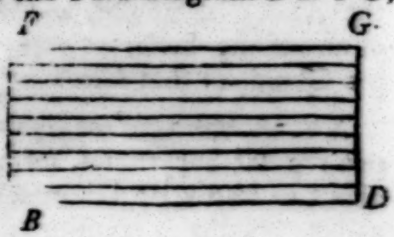
XX. The curve Superficies of all circular Solids (viz. Cylinders, Cones, Spheres, &c.) are composed of an infinite Series of the Peripheries of those Circles which constitute their Solidities.

Upon these Definities are grounded all the following Theorems; and therefore, if they were diligently compared with their respective Figures, it must needs be of great Help to the Learner, and would render all that follows very easy; wherein I shall begin with what hath been already demonstrated, by way of introducing the rest.

THEOREM I.

The Area *of every Right-angled* Parallelogram *is obtained by multiplying the Length into its Breadth.*

That is, $BD \times FB =$ the Area of the Parallelogram $BDFG$, by *Lemma* 1, compared with Definition 1.



Example.

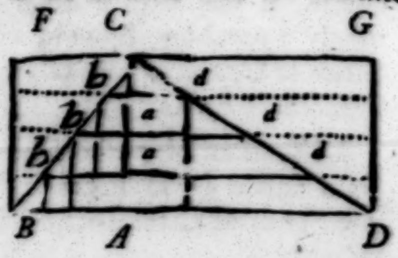
Suppose $BD = 26$, and $FB = 9$, then $26 \times 9 = 234$ the Area. See *Prob.* 1, *Page* 339.

THEOREM II.

The Area *of every plain Triangle is equal to half the* Area *of its circumscribing* Parallelogram, *That is,* $\frac{BD \times BA}{3} =$ *the* Area *of* $\triangle BCD$, *in the following Figure.*

Demonstration.

Suppose the Perpendicular CA to be divided into an infinite Number of equal Parts, as at the Points a, a, a, &c. and through those Points there were drawn Right Lines parallel to the Base BD; (*viz.* bad, bad, bad, &c.) then will those Lines be a Series of Terms in Arithmetical Progression beginning at the Point C (*viz.* 0, b, d, 2 b d, 3 b d, &c. as is evident by the Figure, wherein BD is the greatest Term $= L$, and CA the Number of Terms $= N$.



But

But $\frac{1}{2} NL = S$, by Lemma 2. And $S =$ the Triangle's Area by Definition 2. Q. E. D.

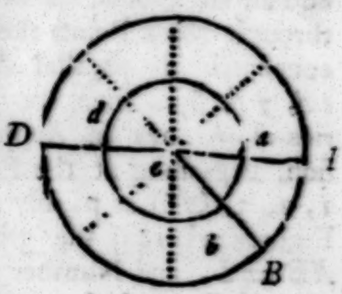
Example. Let $BD = 26$, and $CA = 9$ as above; then $\frac{26 \times 9}{2} = 117$, or $\frac{26}{2} \times 9 = 117$. Or thus $26 \times \frac{9}{2} = 117$ the Area required. [See *Problem* 3, *Page* 330.]

THEOREM III.

The Peripheries of Circles are in Proportion one to another as their Diameters are.

Demonstration.

Let the Periphery of a Circle be divided into any Number of equal Arches by Right Lines drawn from the Center (*viz. Radii*) suppose them 8, as in the annexed Figure, wherein AB is one of them; then, if thro' any Point in the Radius there be drawn a concentric or parallel Circle, its Periphery will also be divided into 8 equal Arches by those *Radii*, one whereof will be ab, and the $\triangle\, Cab$ will be like to $\triangle\, CAB$. Therefore $Ca : ab :: CA : AB$, or $Ca : CA :: ab : AB$, consequently $2\,Ca : 2\,CA :: 8\,ab : 8\,AB$ But $2\,Ca = da$ the Diameter of the Circle, whose Periphery is $8\,ab$; and $2\,CA = DA$, the Diameter of the Circle, whose Periphery is $8\,AB$. Therefore, &c. as by the Theorem. Q. E. D.



Example.

In Chapter 9, Part III, it was found, that, if the Diameter of a Circle be 2, its Periphery will be 6,2831853, &c. Therefore, 2 : 6,2831853, &c. :: 1 : 3,14159265, &c. The Periphery of the Circle whose Diameter is 1.

Corollary.

Hence it follows, that because Unity, or 1, may be made the first Term in the Proportion, therefore 3,14159265, &c. may be made a constant or settled Factor; which, being multiplied into any proposed Diameter, will produce the Periphery of that Circle.

Note, Instead of 3,14159265, &c. it may be sufficient to take only 3,1416.

Or,

Or, in whole Numbers the Proportion may be,

As 7 : 22 : : Diam. : Periphery } { these Numbers may serve,
Or 113 : 355 : : Diam. : Periphery } { and are often used in common Practice.

THEOREM IV.

The Area *of any* Sector *of a* Circle *is equal to half the Rectangle of the* Radius *into its Arch.* That is, $\frac{CA \times AB}{2}$ = the Area of ACB.

Demonstration.

Suppose the Radius CA to be divided into an infinite Series of equidistant Points, as a, e, y, &c. and through those Points there were drawn concentric or parallel Arches, as ab, ed, yf, &c. then they will be a Series of Arches in Arithmetic Progression, beginning at the Point C (*viz.* 0, 1, 2, 3, *&c.*) as plainly appears by the Figure, wherein the greatest Term is $AB = L$, and Number of Terms is $CA = N$. But $\frac{1}{2} NL = S$ the Sum of all the Series, by Lemma 2, and $S =$ the Sector's Area, by Definition 3.

Q. E. D.

Let the *Radius* $CA = 12$, and the Arch $AB = 8$, then $\frac{12 \times 8}{2} = 48$. or $\frac{12}{2} \times 8 = 48$. or $\frac{8}{2} \times 12 = 48$, the Area of the Sector ACB.

THEOREM V.

The Area *of every Circle is equal to half the Rectangle of the* Radius *into its* Periphery. *That is according to* Archimedes, *a Circle is equal to a Right-angled Triangle, whose Sides containing the Right-angle are equal, one to the* Radius, *and the other to the* Perimeter *of that Circle.* Pro. 1. de Dimensione Circuli.

The Truth of this Theorem may be easily deduced from the last thus; If we suppose the last Sector to be one Eighth-part of a Circle, then it follows, that $\frac{8AB \times CA}{2} = 4AB \times CA$ will be the Area of the whole Circle. But $4AB$ = half the Circle's Periphery, and CA = half its Diameter; therefore, &c. as per Theorem.

Q. E. D.

Example.

Example.

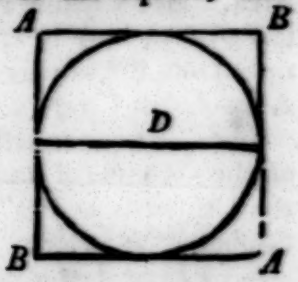
If the Diameter be Unity, or 1, the Periphery will be 1,14159265 &c. by *Theorem* 3. Then $\frac{3,14159265}{2} \times \frac{1}{2} = 0,78539816$, &c. (or 0,7854 for common Use) will be the Area of that Circle.

Scholium.

From hence naturally flows the following Proportion between the Square and its inscribed Circle.

PROPORTION. { As the Perimeter (*viz.* the Sum of the four Sides) of any Square : is to its Area :: so is the Periphery of the inscribed Circle : to its Area.

That is, supposing $AB = D =$ the Side of the Square, and the Diameter of its inscribed Circle; then $4D =$ the Perimeter, $DD =$ the Area of the Square, and $3,1416\,D =$ the Periphery of the Circle, by *Theorem* 3. But $4D : DD :: 3,1416\,D : 0,7854\,DD =$ the Circle's Area. Aud if $D = 1$; then $4D = 4$, and $DD = 1 \times 1 = 1$; and the Periphery will be 3,1416. Then $4 : 1 :: 1 : 0,7854$ &c. as in the *Example* above.



And from hence may be easily deduced the following Theorems.

THEOREM VI.

The Area's *of all Circles are in Proportion one to another as the Squares of their Diameters. (2. e. 12.)*

For if $D =$ the Diameter of one Circle, and $d =$ the Diameter of another Circle, then will $0,7854\,DD$ be the Area of one Circle, and $0,7854\,dd$ will be the Area of the other Circle; as above. But $0,7854\,DD : 0,7854\,dd :: DD : dd$. Or thus, let $D =$ the Diameter, and $P =$ the Periphery of one Circle; $d =$ the Diameter, and $p =$ the Periphery of another Circle;

Then	1	$\frac{1}{2}D \times \frac{1}{2}P = \frac{1}{4}DP = A$, the Area of one Circle.
And	2	$\frac{1}{2}d \times \frac{1}{2}p = \frac{1}{4}dp = a$, the Area of the other Circle.
$1 \times \overline{4}$	3	$DP = 4A$ (per last Theorem.
$2 \times \overline{4}$	4	$dp = 4a$
$2 \div D$	5	$P = \frac{4A}{D}$

4 ÷ d

$4 \div d$	6	$p - \frac{4a}{d}$
But	7	$P : p :: D : d$, per Theorem 3.
5, 6, 7	8	$D : d :: \frac{4A}{D} : \frac{4a}{d}$
8 ∵	9	$4DDa = 4ddA$, that is, $DDa = ddA$
9, *Analogy*	10	$DD : A :: dd : a$, or $A : a :: DD : dd$

Q. E. D.

Corollary.

Hence it follows, that because the Square of 1 is 1 (*viz.* $1 \times 1 = 1$) and 0,785398l6, &c. or 0,7854 is the Area of the Circle whose Diameter is one (as before) therefore it will be 1 : 0,7854 :: so is the Square of any Circle's Diameter : to its Area. And because 1 is the first Term in the Proportion, therefore 0,7854 may be made a constant Factor; which, being multiplied into the Square of any proposed Diameter, will produce the Area of that Circle.

Note, The four last Theorems do plainly shew the Reason of all the common or practical Problems about a Circle, which, for the Learner's farther Satisfaction, I have here inserted together. Supposing as before.

That $\left\{\begin{array}{l} D = \text{the } \textit{Diameter} \\ P = \text{the } \textit{Periphery} \\ A = \text{the } \textit{Area.} \end{array}\right\}$ of any proposed Circle;

		Probl. 1. D *being given, to find* P.
Then	2	$1 : 3,1416 :: D : P$, per Theorem 3.
1 ∵	1	$3,1416\,D = P$
Examp.		Suppose $D = 32$. Then $3,1416 \times 32 = 100,5312$ the *Peripbery*.
		Probl. 2. D *being given, to find* A.
	3	$1 : 0,7854 :: DD : A$, per Theorem 6.
3 ∵	4	$0,7854\,DD = A$
		Suppose $D = 32$ (as before)
Examp.		$DD = 32 \times 32 = 1024$
Then		$0,7854 \times 1024 = 804,2496$, the Area required.
And		*Probl.* 3. P *being given, to find* D.
$2 \div$	5	$D = \frac{P}{3,1416}$ or $\left\{\begin{array}{l}\text{because } \frac{1}{3,1416} = 0,3183 \\ \text{therefore } 0,3183\,P = D.\end{array}\right.$
		This being only Converse to the 1st. needs no Exam.

		Prob. 4. *P* being given, to find *A*.
2 $\bullet^2$	6	$9,86965\,DD = PP$
6 ÷	7	$DD = \frac{PP}{9,86965}$, or $0,10132\,PP = DD$
4 ÷	8	$DD = \frac{A}{0,7854}$, or $1,2732\,A = DD$
For		$\frac{1}{0,7854} = 1,2732$
7, 8	9	$\frac{PP}{9,86965} = \frac{A}{0,7854}$, or $0,10132\,PP = 1,2732\,A$
9 × &c.	10	$\frac{PP}{12,5664} = A$, or $0,07957\,PP = A$
		Prob. 5. *A* being given, to find *D*.
8 ww^2	11	$D = \sqrt{\frac{A}{0,7854}}$, or $D = \sqrt{1,2732\,A}$
		Prob. 6. *A* being given, to find *P*.
10 × &c.	12	$PP = 12,5664\,A$, or $PP = \frac{A}{0,07957}$
12 ww^2	13	$P = \sqrt{12,5664\,A}$, or $P = \sqrt{\frac{A}{0,07957}}$

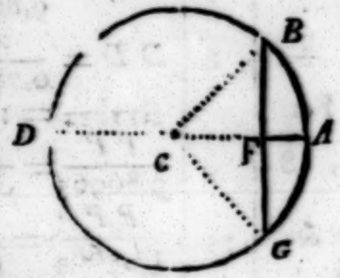
These six Problems contain all the Variety that can be proposed about finding the Periphery, Diameter, and Area of any Circle.

But if it be required to find the Area of any Segment, or Part of a Circle cut off by a Chord, that Work will require a farther Consideration.

First, As to the *Data* there must always be given the *Diameter*; or, either the *Periphery* or *Area* of the *Circle*, in order to find the Diameter.

Secondly, There must also be given, either the *Chord*, which is the Base of the Segment, or the versed Sine, which is the Height of the Segment. That is, either *B G*, or *A F*, in the following Scheme, must be given, that so the *Area* of the △ *BCG* may be found. Then it is evident (by the Figure) that, if the *Area* of the △ *B C G* be taken from the *Area* of the Sector *C B A G*, the Remainder will be the *Area* of the Segment *B A G*. And if the *Area* of the Segment *B A G* be taken from the whole *Area* of the Circle, the Remainder will be the *Area* of the other Segment *D B G*.

Example in Numbers.



Let there be given $DA = 32$. as in *Prob.* 1. and the *versed Sine* $AF = 6$; then $\frac{1}{2}\ DA = BC = CA = 16$, and $CA - AF = CF = 10$. But $\square BC - \square CF = \square BF$. Consequently, $\surd \square BC - \square CF = BF$, *viz.* $\surd 156 = 12{,}49 = BF$.

Then, by the Doctrine of plain *Triangles*, the *Arch* $BA = \angle BCA$ may be found in Degrees and Decimal Parts. Thus BC : *Radius* : : BF : *Sine* $\angle BCF = 51{,}31$ Degrees. And then it will always hold in this Proportion;

Viz. { *As the Circle's Periphery in Degrees* : *is to its Periphery in equal Parts* (according to the Dimensions taken) : : *So is the Arch in Degrees* (viz. $\angle BCA$) : *to the same Arch in equal Parts.*

That is, $360^\circ : 100{,}5312 :: 51^\circ{,}31 : 14{,}3284 = BA$. Then $14{,}3284 \times 16 = 229{,}2544$, the *Area* of the Sector $BCAG$; and $12{,}49 \times 10 = 124{,}9$, the *Area* of the $\triangle BCG$. Their Difference $104{,}3544 =$ the *Area* of the Segment BAG.

Or the *Area* of any Segment may be otherwise found (as most usually it is) by a Table of the Segments of a Circle, whose *Area* is Unity, or 1. The Construction or making of such a Table is very well laid down in Mr. *Darie*'s Book of Gauging, Chap. 9. which he performs in this Problem.

PROBLEM.

In a Circle whose Area is Unity, and its Diameter cut by Chord Lines into 1000 *equal Parts, to find the Segment to any versed Sine proposed, not exceeding* 500 *of those equal Parts.*

1. Multiply the versed Sine proposed by 0,002, and substract the Product from an Unit or 1.

2. This Remainder you shall seek in the common Table of natural Sines, (the Arch being divided into Degrees and Centesimals) which being found, let its *Co-arch* be doubled, and called *A*.

3. You must find the correspondent Sine to *A*; which Sine being found, you may call , and then it holds 6,2831853) 0,017453 2925 $A - S$ (= the Segment required.

Now

Now this Segment being thus found, if you subduct it from an Unit, you have the *Co-segment*, &c.

Note, Notwithstanding what has been said in the second Precept of this Problem, it very often falls out that the Remainder there spoken of cannot be truly found in the Table of natural Sines; therefore in this Case my Advice is, that you make two Operations, one with a Sine the next greater, and one with a Sine the next less, and in so doing you will be sure to have the Segment required bounded between the Results of those two Operations.

Example. *Let it be proposed to find the correspondent Segment to the versed Sine* 263.

First, $263 \times 0{,}002 = 0{,}526$, and $1 - 0{,}526 = 0{,}474$, its *Arch* is $28°{,}29$ being less than just; its Complement is $61°{,}71$, which, being doubled, is $123{,}42 = A$.

Then $0{,}174533\ A = 2{,}154086286$
$-0{,}8346556 = S$ the Sine of A.

6,2831853) 1,319430686 (0,209993 the Segment.

Now I make a second Work.

263 being multiplied with 0,002 is 526. and $1 - 526 = 0{,}474$ its *Arch* is $28°{,}30$ being greater than just; and its Complement is $61°{,}70$, which being doubled is $123{,}4 = A$.

Then $0{,}0174533\ A = 2{,}1537372$
$-0{,}8348478 = S$ the Sine of A

6,2831853) 1,3188894 (0,209907 the Segment.

So you see by these two Operations that the Segment is bounded, and it is very probable it may be 0,20995.

But to abbreviate this large Factor, and this large Divisor, I shall here insert two Tables of them, which will be ready for Use, and exact enough too.

Divisor.	
6,2832	1
12,5664	2
18,8495	3
25,1327	4
31,4159	5
37,6991	6
43,9823	7
50,2655	8
56,5487	9

Factor.	
,0174533	1
,0349066	2
,0523599	3
,0698139	4
,0872665	5
,1047197	6
,1221730	7
,1396263	8
,1570796	9

Thus far, Mr. *Darie*, which I have here inserted to shew the *Learner* how, by the Help of these two *Tables*, and a *Table* of *Natural Sines*, he may easily make a Table of Segments, whose Use shall be shewed farther on, *viz.* when I come to treat of practical Gauging. In the mean Time I shall here lay down another Method to find the *Area* of any Segment of a

Circle (very near) by a new Theorem, without the Help either of a Table of Sines or Segments, having the same *Data* as before in Page 410.

Viz. Let $\begin{cases} R = \text{the } \textit{Radius}, \text{ or } \frac{1}{2} \textit{ Diameter} \text{ of the given } \textit{Circle}. \\ d = \text{the Difference between the } \textit{versed Sine} \text{ and } \textit{Radius}. \\ C = \text{half the } \textit{Chord} \text{ of the Segment's Base.} \end{cases}$

THEOREM. $\left\{ \dfrac{2\frac{1}{3}RR - 1\frac{1}{3}Rd - dd}{1\frac{1}{2}R + d} \times C = S, \right.$ the *Area* of the *Segm.*

Example, Suppose $R = BC = 16$, $d = FC = 10$, and $C = BF = 12{,}49$; as before.

Then $2\frac{1}{3} RR = 597{,}3333$. $1\frac{1}{3} Rd = 213{,}3333$. $dd = 100$
$- 313{,}3333 = 1\frac{1}{3} Rd + dd$

$1\frac{1}{2} R + d = 34$ 284,0000 (8,3529. Lastly, $8{,}3529 \times 12{,}49 = 104{,}3276$ the *Area* of the Segment *BAG*, as before.

THEOREM VII.

As Squares are to the Areas *of their inscribed Circles, so are Parallelograms to the* Areas *of their inscribed Ellipses.*

That is, $\left\{\right.$ As the Square of the Diameter of any Circle : is to its *Area* : : so is the Rectangle of the Transverse and Conjugate Diameters of any Ellipsis : to its *Area*.

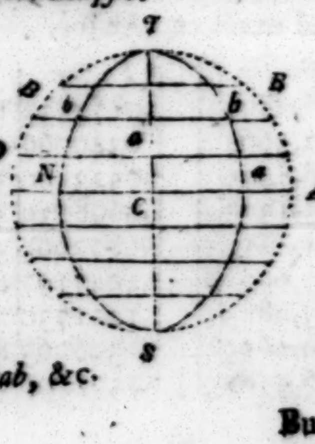
DEMONSTRATION.

Circumscribe any *Ellipsis* with a *Circle*; and suppose an infinite Number of *Chord Lines* drawn therein, all parallel to the *Conjugate Diameter*, as those in the annexed Figure; then it will

be $\left\{\right.$ *As (DA) the Diameter of the Circle : is to (Nn) the Conjugate Diameter of the Ellipsis : : so is (BaB) any Chord in the Circle : to (bab) its respective Ordinate in the Ellipsis.*

For according to the Property of the *Circle*

it is	1	$\overline{TS - Ta} \times Ta = \square\, Ba$
And		by the Property of the *Ellipsis*
it is	2	$\square TC : \square NC :: \overline{TS - TA} \times Ta : \square ba$
1, 2	3	$\square TC : \square NC :: \square BA : \square ba$
3 Hence	4	$TC : NC :: Ba : ba$
Conseq.	5	$2\,TC : 2\,NC :: 2\,Ba : 2\,ba$
That is	6	$DA : Nn :: Bab : bab$
Put	7	$D = 2TC$, and $d = 2NC$
Then	8	$D : d ::$ *Chord* BaB : *Ordinate* bab, &c.



But

But the Sum of an *infinite Series* of such *Chords*, as *B a B*, do constitute the *Area* of the Circle, by Definition 3: and the Sum of the like Series of their respective *Ordinates*, as *b a b*, do constitute the *Ellipsis's Area*, by Definition 4. Therefore $D : d :: $ *Circle's Area* : *Ellipsis's Area*, by *Lemma* 6. But $D : d :: DD : Dd$. Whence it follows, that DD : *Circle's Area* $::$ Dd : *Ellipsis's Area*. Q. E. D. Consequently, as 1 : is to 0,7854 :: so is the Rectangle or Product of the Transverse and Conjugate Diameters of any Ellipsis : to its *Area*.

Example. Suppose $TS=36$. and $Nn=16$; then $36\times16=576$, and $576\times0,7854=452,3904$ the *Area* of the *Ellipsis*.

Corollaries.

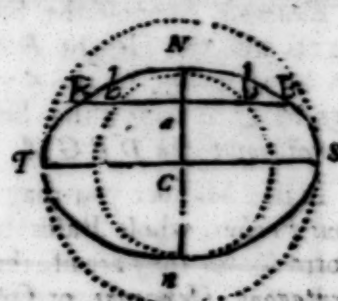
1. Hence it is easy to conceive, that the square Root of the Rectangle or Product of the Transverse and conjugate Diameters will be the Diameter of a *Circle* whose *Area* will be equal to the Ellipsis's *Area*, *viz*. $\sqrt{576}=24$ the Diameter of a Circle $=$ to the Ellipsis.

2. All Segments of an Ellipsis and its *circumscribing Circle* (whose Bases are parallel to the conjugate Diameter, and of the same Height) are in Proportion one to another, as their Bases are. That is, $BaB : bab ::$ *Area* Segment BNB : *Area* Segment bNb; or $TS : Nn ::$ *Area* Segment BNB: *Area* Segment bNb.

THEOREM VIII.

The Area *of every* Ellipsis *is a mean Proportional between the* Areas *of its* circumscribing *and* inscribed Circles.

The Truth of this Theorem may be easily deduced from the last; for supposing $D = TS$, and $d = Nn$, as before; then it is already proved, that $DD : Dd ::$ circumscribing Circle's *Area* : Ellipsis's *Area*. But $DD : Dd :: Dd : dd$. Therefore Ellipsis's *Area* : inscribed Circle's *Area* $:: Dd : dd$. By Theorem 6.



Example. Let $TS = D = 36$. and $Nn = d = 16$, as before; then $DD = 1296$, and $dd = 256$,

Then

Then will $\begin{cases} 1296 \times 0{,}7854 = 1017{,}8784 \text{ the great } \textit{Circle}\text{'s Area} \\ 256 \times 0{,}7854 = 201{,}0624 \text{ the lesser } \textit{Circle}\text{'s Area.} \end{cases}$

Suppose $A =$ the Ellipsis's *Area*; then, according to the Theorem, it will be, $1017{,}8784 : A :: A : 201{,}0624$. *Ergo* $AA = 017{,}8784 \times 201{,}0624 = 204657{,}07401216$. Consequently, $204657{,}07401216 = 452{,}3904 = A$, the *Area* of the *Ellipsis*, as before in the last Example.

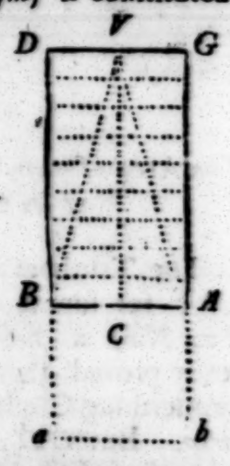
Corollary.

From hence it follows, that all Segments of an Ellipsis and its inscribed *Circle*, whose *Bases* are parallel to the transverse *Diameter*, and have the same Height, are in Proportion one to another as the *Areas* of the Ellipsis and Circle are. That is, *Area* of Circle : *Area* of Ellipsis : : Segment *b N b* : Segment *B N B*. Or, *Nn* : *TS* : : *Area* Segment *b N b* : *Area* Segment *B N B*.

THEOREM IX.

The solid Content of any Prism (what Figure soever its Base is of) *is obtained by multiplying the* Area *of its* Base *into its Height.*

For Instance, a *Parallelopipedon* (or square *Prism*) is constituted of an *infinite Series* of equal *Squares*; that of its Base *BAba* being one of the Terms, and its Height *DB*, or *GA*, the Number of all the Terms. Consequently, the *Area* of $BAba \times DB =$ the Sum of all the Series (by *Lemma* 1.) which is the Solidity of the *Parallelopipedon DBGA*, by Definition 10.



Example. Suppose the Side of the Base $BA = 16$ and the Height $DB = 42$; then will $16 \times 16 = 256$ be the *Area* of the Base, and $256 \times 42 = 10752$ the Solid Content of the *Parallelopipedon DBGA*.

In this Manner you may find the Solidity of all regular *polygonous Prisms*, whose Bases (or Ends) are parallel and alike, what Form soever they are of, that is, whether their *Bases* are *Triangles*, *Pentagons*, *Hexagons*, or *Octagons*, &c.

THEOREM X.

Every Pyramid is the third Part of the Prism, that hath the same Base and Height with it. (7. e. 12.)

That is, the Solid *Content* of the *Pyramid* BVA (in the last Figure) is one third of its circumscribing *Prism* $DBGA$.

Demonstration.

For every Pyramid that hath a square *Base* (as $BAba$, in the last Figure) is constituted of an infinite Series of Squares, whose Sides or Roots are continually encreasing in *Arithmetic Progression*, beginning at the *Vertex* or *Point* V (See *Theor.* 2.) its *Base* $BABa$, being the greatest *Term* ($= LL$) and its *perpendicular Height* VC, or DB, is the Number of all the *Terms* $= N$; but $\frac{NLL}{3} = S$ the Sum of all the Series, by *Lemma* 3, and $S =$ the Solid Content of the *Pyramid* BVA, by Definition 13.

Example, Suppose the Side of a *Pyramid*'s *Base* be $BA = 16$. and its Height be $VC = 42$. Then $16 \times 16 = 256$ the *Area* of its *Base* $BABa$, and $\frac{256 \times 42}{3} = 3584$. Or $\frac{256}{3} \times 42 = 3584$ or thus, $256 \times \frac{42}{3} = 3584$, is the Solidity of that *Pyramid* BVA.

Corollary.

From hence it will be easy to conceive, that every *Pyramid* is $\frac{1}{3}$ of its circumscribing *Prism*, what Form soever its *Base* is of, *viz.* whether it be a *Square*, *Triangle*, or *Pentagon*, &c.

THEOREM XI.

The Solid Content of every Cylinder is obtained by multiplying the Area *of its Base into its Height.*

For every *Right Cylinder* is only a round *Prism*, being constituted of an *infinite Series* of equal *Circles*; that of its *Base* or End being one of the Terms, and its Height BD is the Number of all the Terms. Therefore the *Area* of its *Base* BA, being multiplied into DB, will be its Solidity, by *Lemma* 1. *viz.* Let $D = BA$, and $H = GA$. Then $0{,}7854\ DD \times H =$ its Solidity.

Example.

Example. Let the *Diameter* of its *Base* be $D = 16$, and its Height $H = 42$. Then $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$ the *Area* of its *Base*. And $201,0624 \times 42 = 8444,6208$ the *Solid Content* of that *Cylinder* $DBGA$.

Corollary.

Hence it is evident that every square *Parallelopipedon* is to its inscribed *Cylinder*, as $1 :$ is to $0,7854$. Or in whole Numbers, as $452 :$ to 355 very near. And that all *Prisms* are in Proportion to their inscribed Cylinders, as the *Areas* of their *Bases* are.

THEOREM XII.

The Curve Superficies of every Right Cylinder is equal to the Rectangle made of its Height into the Periphery of its Base.

That is, DB, multiplied into the Periphery of the *Diameter* BA, will produce the *Curve Superficies* of the last *Cylinder* $DBGA$. For the *Cylinder* is constituted of an *infinite Series* of equal *Circles* (according to the last Theorem.) Therefore its Curve Superficies is composed of the *Peripheries* of those *Circles*, by Definition 20. But the *Periphery* of its *Base* BA is one of the *Terms*, and its Height DB is the Number of Terms. Therefore, &c. as by *Lemma* 1. To which, if there be added the *Areas* of both its Ends (or *Bases*) the Sum will be the Superficies of the whole *Cylinder*.

Example. Suppose the Diameter of its Base to be $BA = 16$, and its Height $DB = 42$; as before, then $1 : 3,1416 :: 16 : 50,2656$ the *Periphery* of its *Base*. Again, $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$ the *Area* of each End or Base.

Then $50,2656 \times 42 = 2111,1552$ the *Curve Superficies*, to which add $201,0264 \times 2 = 402,1248$ both the End *Areas*.

The *Sum* $= 2513,2800$ is the *Superficies* of the whole *Cylinder*.

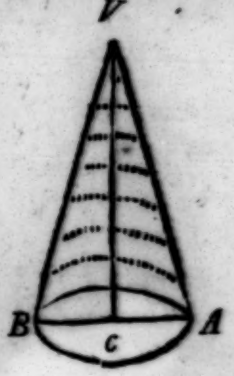
THEOREM XIII.

Every Cone is the third Part of a Cylinder, having the same Base with it, and their Altitudes equal. (10. *e.* 12.)

DEMON-

Demonstration.

The Truth of this Theorem may be easily conceived by only considering, that a Cone is but a round Pyramid, and therefore it must needs have the same Ratio to its circumscribing *Cylinder* as the square Pyramid hath to its circumscribing *Parallelopipedon*, viz. as 1 : to 3. However, to make it yet clearer, let it be farther considered, that every Right Cone is constituted of an infinite Series of Circles, whose Diameters do continually encrease in Arithmetical Progression beginning at the *Vertex* or *Point V*, the *Area* of its *Base B A* being the greatest Term, and its perpendicular Height *V C* the Number of all the Terms; therefore the *Area* of the Circle $BA \times \frac{1}{3} VC$ will be the Sum of all the Series, by Lemma 3, which is the Cone's Solidity.



Example. Let the Diameter of its *Base* be $BA = 16$, and its Height $VC = 42$; Then $1 : 0{,}7854 :: 16 \times 16 = 256 : 201{,}0624$ the *Area* of the *Base*; and $\frac{201{,}0624 \times 42}{3} = 2814{,}8736$ the *Solidity* of the *Cone* *B V A*. Or thus, $201{,}0624 \times \frac{42}{3} = 2814{,}8736$, &c.

Corollary.

Hence it follows that every square Pyramid is to its inscribed Cone as 1 : 0,7854. (Or as 452 : 355) consequently, that all Pyramids have the same *Ratio* to their inscribed *Cones* as the *Areas* of their Bases have.

THEOREM XIV.

The Curve Superficies of every Right Cone is equal to half the Rectangle of the Periphery of its Base into the Length of its Side.

The Truth of this Theorem is self-evident from the Definition of a Cone, Chap. 1. Part IV, where it appears that the Curve Superficies of every Right Cone (as *BVA*) is equal to the Area of a Sector of that Circle whose *Radius* is the Side of the Cone (*VB*) and its Arch equal to the Periphery of the Cone's Base (*B A*). But the Area of any Sector is equal to half the Rectangle of the Radius into its Arch, by Theorem 4. Therefore, &c.

 Exam-

Example. Suppose the Length of the Cone's Side to be *VB*, or *V A* = 42,7551, and the Diameter of its Base, *viz*. *B A* = 16 (as before) then will 50,2656 be the Periphery of its Base, and $\frac{50,2656 \times 42,7551}{2} = 1074,5553$, &c. the Curve of the *Superficies*; to which if there be added the Area of its Base, the Sum will be the *Superficies* of the whole (viz. *all the*) Cone.

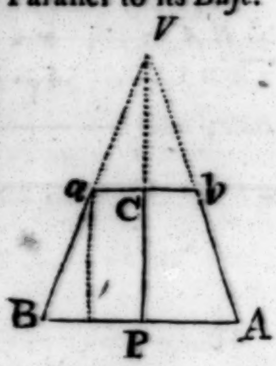
That is 1074,5553
+ 201,0624 the Area of the Base.

Sum 1275,6177 is the total *Superficies*, &c.

Note, *The Truth of this Theorem may be proved from the Consideration of the last Theorem, and Definition* 20.

Scholium.

From the 10th and 13th *Theorems* may be easily deduced several *Theorems* for finding the solid Content of any *Frustum* or *Part* either of a Pyramid or Cone, cut by a plain Parallel to its *Base*. Suppose a square Pyramid, as *BVA*, to be cut by a Plane at *a b*, parallel to its *Base* *B B*, and it were required to find the *Solidity* of the *Frustum* or Part *a b AB*; let there be given *D* = *B A* the Side of the greater Base. *d* = *b a* the Side of the lesser Base. *H* = *C P* the perpendicular Height.



First,	1	$D-d : H :: d : \frac{dH}{D-d} = VC$ by the Figure.
Then	2	$\left\{ DD \times \frac{H+VC}{3} \right.$ = the whole Pyramid *BVA*. By *Theorem* 10
And	3	$dd \times \frac{1}{3} VC$ = the Pyramid *a V b* cut off.
Viz. 1,2	4	$\left\{ \frac{DDDH}{3D-3d} \right.$ = the whole Pyramid *BVA*.
And 1,3	5	$\left\{ \frac{dddH}{3D-3d} \right.$ = the Pyramid *a V b*.
4 — 5	6	$\left\{ \frac{DDDH - dddH}{3D-3d} \right.$ = the Frustum *a b A B*.
6. *Reduc*.	7	$\overline{DD + Dd + dd} \times \frac{1}{3} H$ = the Frustum *a b A B*.

Which in Words gives this following *Theorem*.

THEO-

THEOREM XV.

To the Rectangle of the Sides of the two Bases, *add the Sum of their* Squares; *that Sum, being multiplied into one third of the* Frustum's *Height, will give its Solidity.*

Example. Suppose the Side of the greater Base $BA = 16$ and the Side of the lesser Base (or Top) $ab = 12$ the Height $CP = 9$. Then $16 \times 12 = 192$. $16 \times 16 = 256$. and $12 \times 12 = 144$. Next $192 + 256 + 144 = 592$. and $\frac{592 \times 9}{3} = 1776$. Or $592 \times \frac{1}{3} = 1776$ the Content of the *Frustum* of a square *Pyramid.*

And if it were the like *Frustum* of a Right Cone, it may be found by the same *Theorem.* Supposing $D =$ the Diameter of the greater Base, $d =$ the Diameter of the lesser, and $H =$ the Height of the *Frustum,* then the Sum of all the Squares which constitute the *Frustum* of a square *Pyramid,* are to the Sum of all the Circles which constitute the like *Frustum* of a right Cone, in the Ratio of 1 : to 0,7854 (or of 452 : to 355) therefore it will be $1 : 0{,}7854 :: \overline{DD + Dd + dd} \times \frac{1}{3} H : \overline{0{,}7854\ DD + 0{,}7854\ Dd + 0{,}7854\ dd} \times \frac{1}{3} H =$ the *Cone*'s Frustum, that is, in the last Example, $1 : 0{,}7854 :: 1776 : 1394{,}8704$ the like Frustum of a right Cone. Or because $\frac{1}{0{,}7854} = 1{,}273236$, &c. Therefore it may be made $1{,}273236) \overline{DD + Dd + dd} \times \frac{1}{3} H$ (= the same *Frustum*; that is, $1{,}273236)\ 1776\ (1394{,}17$, &c. as before. And if you take the *Triple* of this Divisor, viz. $1{,}273236 \times 3$, it will be $3{,}8197) \overline{DD + Dd + dd} : \times H$ (= the *Frustum,* &c.

Again,

Suppose	1	$x = D - d$, and $F =$ the *Frustum*
Then	2	$DD + Dd + dd = \frac{3F}{H}$, by the 7th Step of the last
$1 \circledcirc 2$	3	$xx = DD - 2Dd + dd$
$2 - 3$	4	$3Dd = \frac{3F}{H} - xx$
$4 \div 3$	5	$Dd = \frac{F}{H} - \frac{1}{3} xx$, or $Dd + \frac{1}{3} xx = \frac{F}{H}$
$5 \times H$	6	$\overline{Dd + \frac{1}{3} xx} \times H = F$ the *Frustum* $abAB$.

Hence we have another easy Theorem for finding the same *Frustum.*

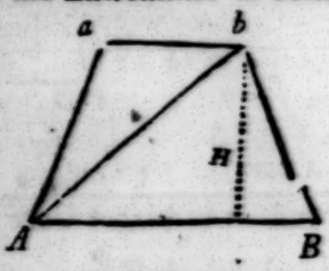
THEOREM XVI.

To the Rectangle of the Sides of the two Bases, add one third Part of the Square of their Difference; that Sum, being multiplied into the Height, will produce the Solidity.

Example. Let $D = 16$. $d = 12$. and $H = 9$, as before; then $Dd = 192$. $D - d = 4 = x$. $\frac{1}{3}xx = \frac{4 \times 4}{3} = 5,3333$, 9997 the Solidity of the Frustum of the square Pyramid, as before. And 3,81968) 1775,9997 (1394,87 &c. the like Frustum of a right Cone, as before.

Either of the two last Theorems (being rightly applied) will produce the true solid Content of all Frustums of any kind of Pyramids, that are intercepted between two parallel and alike Planes or Bases: as above.

But if such Frustums are cut through the Extremities of both Bases by a Diagonal Plane (as *Ab* in the annexed Figure) into two Parts, *Aab* and *ABb*, called *Hoofs*; then the Solidity of those Hoofs is usually found by dividing the middle Term Dd of the Equation $DD + Dd + dd$ into two Parts, and adding one of those Parts to the Square of each Base. Thus,



$\overline{DD + \frac{1}{2}Dd} \times \frac{1}{3}H =$ the great Hoof ABb, and $\overline{dd + \frac{1}{2}Dd} \times \frac{1}{3}H$ = the lesser Hoof Aab of the Frustum of any square Pyramid. Then 3,8197) $\overline{DD + \frac{1}{2}Dd \times H}$ (=) the greater Hoof of a Cone. And 3,8197) $\overline{dd + \frac{1}{2}Dd \times H}$ (=) the lesser Hoof, &c.

These are the Theorems made use of by Mr. *Darie*, in his Book of Gauging, and are pretty near the Truth, but not exactly so; for they give the Solidity of the upper Hoof Aab a small Matter too big, and the lower Hoof ABb as much too little.

Now, in order to rectify that small Error, I shall here propose the two following Theorems, which come very near the Truth, and are more easily performed than those proposed in the first Impression of this Book.

First,

First $\overline{DD+\frac{1}{2}Dd+D-d}\times\frac{1}{3}H$ will be the Solidity of the greater Hoof ABb.

Secondly, $\overline{dd+\frac{1}{2}Dd+d-D}\times\frac{1}{3}H$ will give the Solidity of the lesser Hoof Aab, of the Frustum of any square Pyramid.

And for the like Hoofs of the Frustum of any right Cone, it will be

Thus, 3,8197) $\overline{DD+\frac{1}{2}Dd+D-d}\times$ * H (= the greater Hoof.
And 3,8197) $\overline{dd+\frac{1}{2}Dd+d-D}\times$ H (= the lesser Hoof.

Note, *In order to avoid many Words in the following* Demonstrations, *let* ⊙ *signify any* Circle *in general; and if any two Letters be joined to it, thus,* ⊙ *B A, &c. it then denotes the* Area *of such a Circle as those two Letters represent the* Radius *of.*

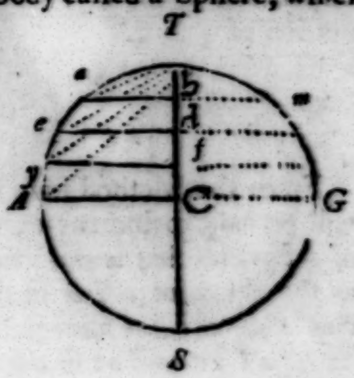
THEOREM XVII.

The Superficies of every Sphere (or Globe) *is equal to four Times the* Area *of its greatest Circle.*

That is, of a Circle whose Diameter is the *Axis* of the Sphere.

Definitions.

If any Circle (as $ATGS$) be turned or moved about its Diameter (TS) it will describe a solid Body called a Sphere, which will be constituted of an infinite Series of concentric or parallel Circles, whose Diameters are Chords, *viz.* ⊙ ab, ⊙ ed, ⊙ ef, &c. by Definition 14. Consequently, the Superficies of the Sphere will be composed of the Peripheries of those Circles which constitute its Solidity, by Definition 20.



Let $D = TS$, the Axis of any Sphere. Then, according to the Property of a Circle, it

will be	1	$\overline{D-Tb}\times Tb=\square ab$
That is	2	$D\times Tb-\square Tb=\square ab$
Therefore	3	$D\times Tb=\square aT$, for $\square ab+\square Tb=\square aT$.
And	4	$D\times Td=\square eT$
	5	$D\times Tf=\square yT$, &c.

* The Error is here corrected, which Mr. *J. Robertson* takes Notice of in his Book entitled, *A Compleat Treatise of Mensuration*, Page 160.

Hence

Hence it is evident, that the Series $\square aT$, $\square eT$, $\square yT$, &c. are in the same Ratio with Tb, Td, Tf, &c. viz. in Arithmetical Progression: Whence it follows, that the $\odot\, aT =$ the Sum of all the Circle's Peripheries between T and b, and $\odot\, eT =$ the Sum of all the Circle's Peripheries between T and d, &c. Consequently, that the $\odot\, AT =$ the Sum of all the Circle's Peripheries included between T and B; that is, $\odot\, AT =$ the Superficies of the *Hemisphere*. And because $\square AC + \square TC = \square AT$, and $\square AC = \square TC$. Therefore $\odot\, AT = 2 \odot AC$ is the Superficies of the Hemisphere. Consequently, $4 \odot AC$ will be the Superficies of the whole Sphere. Q. E. D.

Example. Suppose the Axis $TS = D = 16$. Then $DD = 256$ And $1 : 0{,}7854 :: 256 : 201{,}0624 = \odot\, AC$, for $\frac{1}{2} D = AC$. Then $201{,}0624 \times 4 = 804{,}2496$, the Superficies of the whole Sphere. Or, because $3{,}1416$ is four Times $0{,}7854$, therefore it will always be $1 : 3{,}1416 :: DD : 3{,}1416\, DD$ the Superficies of the Sphere (as before); and it is equal to the curve Superficies of the right Cylinder, whose Diameter and Height are each $=$ D the Axis of the Sphere. For $3{,}1416\, D =$ the Periphery of the Cylinder's Base, and that, multiplied with D its Height, will be $3{,}1416\, DD$ the curve Superficies of the Cylinder, by Theorem 12. And if to this there be added the Area of its two Bases (or Ends) viz. $1{,}5708\, DD$, then it is evident, that the whole Superficies of the Cylinder will be to that of the Sphere in Proportion of 3 to 2.

Scholium.

From the Method here used in proving the last Theorem, it will be easy to find the curve Superficies of any Segment or Part of a Sphere that is cut off by a Right Line or Plane, viz. such as the Segment aTm in the last Scheme, whose curve Superficies is $\odot\, aT$ (as above). Therefore (because $\square ab + \square Tb = \square aT$) it will be $\odot\, ab + \odot\, Tb =$ the curve Superficies of that Segment.

But if the Axis TS, and Height Tb, of the Segment are given, then will it be $TS \times Tb = \square aT$; as in the Third Step above. Which gives this Proportion or Theorem;

Viz.

Viz. { *As the* Axis *of the Sphere* : *is to the whole* Superficies *of the Sphere* : : *so is the* Height *of any Segment* : *to its curve* Superficies.

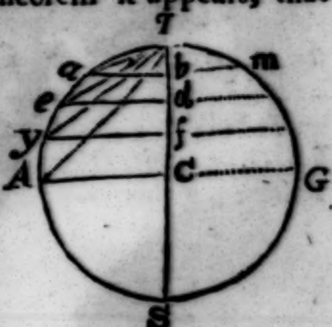
To which if there be added the Area of the Segment's Base, the Sum will be the Superficies of the whole Segment.

THEOREM XVIII.

Every Sphere is equal to two Thirds of its circumscribing Cylinder.

That is, of a Cylinder whose Height and Diameter of its Base are each equal to the Axis of the Sphere.

Demonstration.



According to the Work in the last Theorem it appears, that $\odot\, ab$, $\odot\, ed$, $\odot\, yf$, &c. do constitute the Solidity of the Sphere; and that $\square\, aT$, $\square\, eT$, $\square\, yT$, &c. are a Series of Terms in Arithmetical Progression, $\square\, AT$ being the greatest Term, and TC the Number of Terms; therefore $\odot\, AT \times \frac{1}{2} TC =$ the Sum of all the Series, per *Lemma* 2. And because $\square\, aT - \square\, Tb = \square\, ab$, $\square\, eT - \square\, Td = \square\, ed$, $\square\, yT - \square\, Tf = \square\, yd$, $\square\, AT - \square\, TC = \square\, AC$, &c. wherein $\square\, Tb$, $\square\, Td$, $\square\, Tf$, &c. are a Series of Squares whose Roots Tb, Td, Tf, are in Arithmetical Progression, $\square\, TC$ being the greatest Term, and TC the Number of Terms; therefore $\odot\, TC \times \frac{1}{3} TC =$ the Sum of all that Series, per Lemma 3. consequently, $\odot\, AT \times \frac{1}{2} TC - \odot\, TC \times \frac{1}{3} TC =$ the Sum of the Series $\odot\, ab$, $\odot\, ed$, $\odot\, yf$, &c. which constitute the Solidity of the half Sphere ATG. Put $D = 2TC$ the Axis of the Sphere; then $\frac{1}{4} D = \frac{1}{2} TC$, and $\frac{1}{6} D = \frac{1}{3} TC$. And because $\square\, AT = 2 \square\, TC$; therefore $\odot\, AT = 2 \odot\, TC = 1{,}5708\, DD$. And $1{,}5708\, DD \times \frac{1}{4} D = 0{,}3927\, DDD$.

Again, $\odot\, TC \times \frac{1}{3} TC = 0{,}7854\, DD \times \frac{1}{6} D = 0{,}1309\, DDD$, then $9{,}3927\, DDD - 0{,}1309\, DDD = 0{,}2618\, DDD$ the Solidity of the Semi-sphere ATG, consequently, $0{,}2618\, DDD \times 2 = 0{,}5236\, DDD$ will be the solid Content of the whole Sphere, which is equal to two Thirds of the Cylinder whose Diameter of its Base and Height $= D$. For $0{,}7854\, DDD =$ the Solidity of the Cylinder, by Theorem 11. But $\frac{2}{3}$ of $0{,}7854\, DDD\; 0{,}5236\, DDD$; as before. Therefore, &c. : as by Theorem.

Example.

Example. Suppose the Axis $D=16$, then $DDD=4096$, and $1 : 0,5236 :: 4096 : 2144,6656$ the solid Content of that Sphere.

Corollaries.

1. Hence it appears, that the solid Content of every Sphere is equal to its Superficies multiplied into one sixth Part of its Axis. For its Superficies is $3,1416DD$, by Theorem 17. But $3,1416 \times \frac{1}{6} D = 0,5236\, DDD$ the solid Content, as before.

2. And hence it is also evident, that there is the like Ratio or Habitude between the Cube and its inscribed Sphere, as is betwixt the Square and its inscribed Circle; and that is, as the Superficies of any Cube : is to the Superficies of its inscribed Sphere :: so is the solid Content of that Cube : to the solid Content of the Sphere. [See the Circle's Proportion, Page 407.] For if D = the Side of the Cube, then $6DD$ = its Superficies, and DDD = its Solidity; and $3,1416\ DD$ = the Sphere's Superficies. But $6\,DD : 3,1416\,DD :: DDD : 0,5236\ DDD$ the Solidity of the Sphere; as above.

Scholium.

From the Proof of this Theorem it will be easy to deduce or raise Theorems for finding the solid Content of any Frustum or Segment of a Sphere; as $a\,T\,m$ in the last Figure. For we there suppose the Segment $a\,T\,m$ to be constituted of an infinite Series of Circles, which have the same Ratio with all those Circles that constitute the Semi-sphere. Therefore it follows, that $\odot\, at \times \frac{1}{2} T b - \odot\, bT \times \frac{1}{3} T\, b$ will be the Sum of all the Circles intercepted between T and b. Consequently it will be the Solidity of that Segment. And because $\square\, a\, b + \square\, T\, b = \square\, a\, T$: therefore $\overline{\odot\, a\, b + \odot\, T\, b} \times \frac{1}{2} T b - \odot\, T\, b \times \frac{1}{3}\, b =$ the same Solidity.

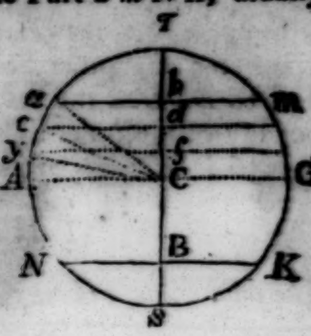
Let $t = a\,b$ half the Segment's Base; $b = Tb$ its Height; and $S =$ the Solidity of the Segment or Frustum: Then $\odot\, ab = 3,1416\, cc$, and $\odot\, T\,b = 3,1416\, bb$. Consequently,

$$\frac{3,1416ccb + 3,1416bbb}{2} - \frac{3,1416bbb}{3} = S$$ which being reduced will become $\overline{3\,ccb + bbb} \times 0,5236 = S$. Or $1,909855)\ 3\,ccb + bbb\ (= S$. for $0,5236)\ 1,0000\ (1,909855$. Which is one Theorem for finding the Frustum's Solidity.

Note,

Note. Here we suppose the Height of the Segment and the Diameter of its Base to be given; but if the Axis of the Sphere, and the Height of the Segment be given, then putting $D =$ the Sphere's Axis, $b =$ the Segment's Height, and c as before, it will be $\overline{D-b} \times b = cc$, viz. $Db - bb = cc$. Therefore $3\,Dbb - 2\,bbb = 3\,ccb + bbb$. Consequently $3\,D\overline{bb - 2bbb} \times 0{,}5236 = S$, the Frustum's Solidity. Or $1{,}90985)\cdot 3\,Dbb - 2\,bbb\ (= S$, as before. Which is a second Theorem for finding the same Frustum aTm.

And if it be required to find the middle Part $amNK$, usually called the middle *Zone* of a Sphere, then because it is supposed that $am = NK$, or which is all one, that $bC = CB$, therefore it is plain, that, if twice the Segment aTm be taken from the Solidity of the whole Sphere, there will remain the middle *Zone* $amNK$. But, because that Work is a little troublesome, I shall here shew how to raise a Theorem for the doing it.



First, Because $AC = yC = eC = aC = TC$. Therefore it will be $\square AC - \square Cf = \square yf$. $\square AC - \square Cd = \square ed$. $\square AC - \square Cb = \square ab$, &c. Here because $\square AC . \square AC . \square AC$, &c. are a Series of Equals, and Cb the Number of all the Terms, therefore $\square AC \times Cb =$ the Sum of all that Series, by Lemma 1. And $\square Cf . \square Cd . \square Cb$, &c. being a Series of Squares whose Roots are in Arithmetical Progression, beginning at the Center or Point C, viz. 0, Cf, Cd, Cb, &c. wherein the greatest Term is $\square Cb$, and Number of Terms is Cb. *Ergo* $\square Cb \times \frac{1}{3} Cb =$ the Sum of all the Series, by Lemma 3. Consequently, the $\odot AC \times Cb - \odot Cb \times \frac{1}{3} Cb =$ the Sum of all the Series $\odot yf . \odot ed . \odot ab$, &c. which do constitute the Solidity of the half Zone $amAG$. And because $\square AC - \square Cb = \square ab$. *Ergo* $\odot AC - \odot ab = \odot Cb$. Consequently $\odot AC \times Cb -$

$$\frac{\overline{\odot AC - \odot ab \times Cb}}{3} = 2 \odot AC + \odot ab \times \tfrac{1}{3} Cb$$ will be the Solidity of the half Zone.

Put $D = AG = 2\,AC . x = am$. and $H = bB = 2\,Cb$. Then $\odot AC = 0{,}7854\,DD . \odot ab = 0{,}7854\,xx$. And if we turn the common Factor $0{,}7854$ into the Divisor $1{,}27323$,

 and

and then take the Triple of that Divisor, viz. 3.8197 (as before in the Frustum of Pyramids)- the Result of the precedent Work will produce this following Theorem.

THEOR. XIX. $\left\{\frac{2DD+xx}{3,8197} : \times H = \right\}$ the middle Zone $amNK$.

THEOREM XX.

Spheres *are in Proportion one to another as the* Cubes *of their Diameters.* (18. *e.* 12.)

Demonstration.

Suppose $D =$ the Diameter or Axis of any Sphere, and $d =$ the Diameter of another Sphere, either greater or lesser. Then is 0,5236 $DDD =$ the Solidity of one Sphere, and 0,5236$ddd =$ the Solidity of the other Sphere, by Theorem 18. But $DDD : ddd ::$ 0,5256 $DDD :$ 0,5236 ddd. Q. E. D.

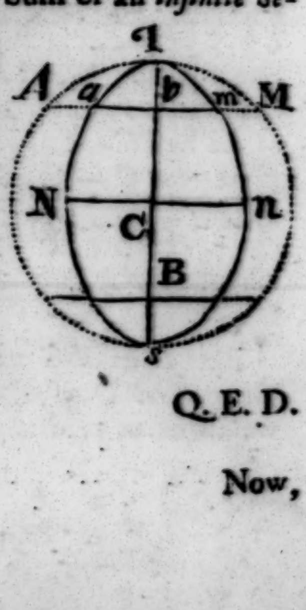
THEOREM XXI.

The solid Content of every Spheroid *is equal to two Thirds of its circumscribing* Cylinder.

Demonstration.

Suppose the Figure $NTnSN$ in the annexed Scheme, to represent a Spheroid, formed by the Rotation of the Semi-Ellipsis TNS, about its Transverse Axis TS (as by Definition 15.)

Let $D = TS$, the Length of the Spheroid, and the Axis of its circumscribing Sphere; and $d = Nn$, the Diameter of the greatest Circle of the Spheroid. Then because $\square\, TC : \square\, NC :: \square\, Ab : \square\, ab$, by Step 3 in Theor. 7, therefore it will be $DD : dd :: \square\, Ab : \square\, ab :: \odot\, Ab : \odot\, ab$, &c. But the Sum of an *infinite Series* of such Circles as $\odot\, Ab$ (whose Diameters are Chords) do constitute the Solidity of the Sphere (as before at Theorem 18) and the Sum of an infinite Series of such Circles as $\odot\, ab$ (viz. whose Diameters are Ordinates of the Ellipsis) do constitute the Solidity of the *Spheroid*, by Defi. 15. *Ergo* $DD : dd ::$ 0,5236 $DDD :$ 0,5236 $ddD =$ the Solidity of the *Spheroid*, by Lemma 6. But 0,5236$ddD = \frac{2}{3}$ of the Cylinder whose Diameter is $= d$, and Height $= D$, by Theorem 11.



Q. E. D.

Now,

Now, from this Proportion between the Sphere and its inſcribed Spheroid, it will be very eaſy to deduce Theorems for finding the ſolid Content either of the Segment or middle Zone of any Spheroid, having the ſame Height with that of the Sphere.

For $\left\{\begin{array}{l}\textit{As the Solidity of the whole Sphere : is to the Solidity of the}\\ \textit{whole Spheroid : : ſo is any Part of the Sphere : to the like}\\ \textit{Part of the Spheroid, by the Converſe to Lemma 6.}\end{array}\right.$

As for Inſtance; ſuppoſe it were required to find the middle Zone of any Spheroid: Let $D=TS$, and $d=Nn$, as above; and $H=bB$, $x=AM$, as in Theorem 19, and let $c=am$. Then $\frac{2DD+xx}{3,8197}\times H=$ the middle Zone of the Sphere. And $0,5236DDD$ $:0,5236\,ddD::\frac{2DD+xx}{3,8197}\times H:\frac{2dd\times H}{3,8197}+\frac{xx\,dd\times H}{3,8197DD}=$ the middle Zone of the Spheroid.

Again, $DD:dd::xx:cc$, therefore $\frac{xxdd}{DD}=cc$. conſequently, $\frac{xxdd}{DD}\times\frac{H}{3,8197}=\frac{cc}{3,8197}\times H$, which being taken inſtead of $\frac{xxdd\times H}{3,8197DD}$, there will ariſe this following

THEOREM XXII. $\left\{\frac{2dd+cc}{3,8197}:\times H=\right\{$the middle Zone of the Spheroid being the very ſame with Theorem 19.

Note, *In the ſame Manner you may raiſe Theorems for finding the Segment of a Spheroid, cut off at either of its Ends, &c.*

THEOREM XXIII.

The Area of every Parabola is equal to two Thirds of its circumſcribing Parallelogram.

Demonstration.

Let the Figure SAB repreſent half a Parabola. Make DB parallel to the Axis SA, and Sd parallel to the Semi-Ordinate AB, and ſuppoſe Sd to be divided into an infinite Series of equidiſtant Points, as f, g, b, &c. and from thoſe Points imagine a

Series of parallel Lines, viz. fm, gn, bp, &c. to touch the Curve of the Parabola, and meet the Semi-ordinates ma, ne, yp, &c. Then, according to the Property of the Parabola, it will

be	1	$SA : \square AB :: Sa : \square am$
	2	$SA : \square AB :: Se : \square en$
	3	$SA : \square AB :: Sy : \square yp$, &c.
But		$Sa = fm$. $Se = gn$. $Sy = bp$. $SA = dB$
		Therefore alternately it will be
3,	4	$\square AB : dB :: \square yp : bp$
2,	5	$\square AB : dB :: \square en : gn$
1,	6	$\square AB : dB :: \square am : fm$, &c.

In these Proportions $\square am$, $\square en$, $\square yp$, &c. are a Series of Squares whose Roots Sf, Sg, Sb. &c. are in Arithmetical Progression, beginning at the Point S. And because the Lines bp, gn, fm, &c. have the same Ratio, therefore they are as such a Series of Squares, wherein dB is the greatest Term, and Sd the Number of Terms. Consequently $\frac{dB \times Sd}{3}$ = the Sum of all those Lines, by Lemma 3. But $SA \times AB = dB \times Sd$. Therefore $\frac{SA \times AB}{3}$ = the Sum of all that Series of Lines; but all those Lines do constitute the Area of the Semi-Parabola's Complement, viz. the Area of what half the Parabola SAB wants of compleating or filling up the Parallelogram $SdAB$. Wherefore $SA \times AB - \frac{1}{3} SA \times AB = \frac{2SA \times AB}{3}$ will be the Area of half the Parabola SAB. Consequently, $\frac{2}{3} SA \times bB$ will be the Area of the whole Parabola bSB. Q. E. D.

Example. Suppose the Base, or greatest Ordinate, of a Parabola to be $bB = 24$, and its intercepted Diameter (or Axis) be $SA = 33$; then $2\,SA \times bB = 66 \times 24 = 1584$. and 3) 1584 (528 the Area of that Parabola.

THEOREM XXIV.

Every Parabolic Conoid is equal to one Half of its circumscribing Cylinder.

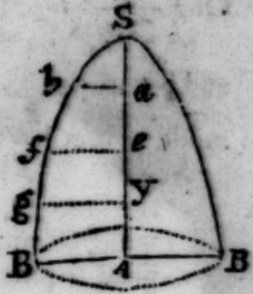
De-

Demonstration.

If any Semi-Parabola (as BSA) be turned or moved about its Axis (SA) it will form a solid Parabolic Conoid, constituted of an infinite Series of Circles, viz. $\odot\, ba$, $\odot\, fe$, $\odot\, gy$, &c. by Definition 17.

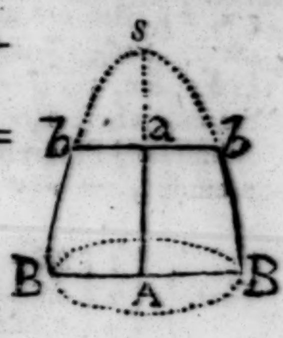
Now, according to the Property of every Parabola, it will be, $SA : AB :: AB : \frac{\square AB}{SA} = L$, the *Latus Rectum*.

$$\text{Then} \begin{cases} Sa \times L = \square\, ba \\ Se \times L = \square\, fe \\ Sy \times L = \square\, gy, \text{ \&c.} \end{cases}$$



Here $Sa \times L$, $Se \times L$, $Sy \times L$, &c. are a Series of Terms in Arithmetical Progression: therefore $\square\, ba$, $\square\, fe$, $\square\, gy$, &c. are also a Series of Terms in the same Progression, beginning at the Point S; wherein $\square\, AB$ is the greatest Term, and SA the Number of all the Terms. Therefore $\square\, AB \times \frac{1}{2} SA =$ the Sum of all the Series by Lemma 2. Consequently, $\odot\, AB \times \frac{1}{2} SA =$ the Sum of all the Series $\odot\, ba$, $\odot\, fe$, $\odot\, gy$, &c. which do constitute the Solidity of the Conoid. And putting $D = 2\,AB$, and $H = SA$, Then $0{,}7854\, DD \times \frac{1}{2} H = 0{,}3927\, DDH$ will be the solid Content of the Conoid, which is just half the Cylinder whose Base $= D$ and Height $= H$. [See Theorem 11,] Q. E. D.

This being understood, it will be easy to raise a Theorem for finding the lower Frustum of any Parabolic Conoid. For supposing $b = aA$ the Height of the Frustum, and $p = Sa$ the Height of the Part bSb cut off; then $b + p = SA$, the Height of the whole Conoid. Consequently, $\frac{\odot AB \times b + \odot AB \times p}{2} =$ Solidity of the whole Conoid. And $\frac{\odot ba + p}{2} =$ the Solidity of the Part cut off.



Ergo	1	$\frac{\odot AB \times b + \odot AB \times p - \odot ba + p}{2} =$ the Solidity of the Frustum.
But	2	$b+p : \square\, AB :: p : \square\, ba$
Conseq.	3	$b+p : \odot\, AB :: p : \odot\, ba$
3 ∴	4	$\odot\, AB \times p = \odot\, ba \times b + \odot\, ba \times p$

4—⊙

$4 - \odot ba \times p$	5	$\odot AB \times p - \odot ba \times p = \odot ba \times b$
1×2	6	$\odot AB \times b + \odot AB \times p - \odot ba \times p = 2F$
$6 - 5$	7	$\odot AB \times b = 2F - \odot ba \times b$
$7 + \odot ba \times b$	8	$\odot AB \times b + \odot ba \times b = 2F$
$8 \div 2$	9	$\dfrac{\odot AB + \odot ba}{2} \times b = F$ the Fruſtum's Solidity.

Let $D = 2AB$, as before, and $d = 2ba$ the Diameter of the Part cut off; then we ſhall have this following

THEOREM XXV. $\left\{ \overline{0{,}3927\,DD + 0{,}3927\,dd} \times b = \right.$ the Solidity of the Fruſtum required.

Or $\left\{ \dfrac{DD + dd}{2{,}5464} \times b = \right.$ the Fruſtum; for, 3927) 1,0000(=2,5464

and becauſe $2{,}5464 + \dfrac{2{,}5464}{2} = 3{,}8196$: therefore it may be made 3,8196) $\overline{DC + dd} \times \frac{1}{2} b$ (= the ſame Fruſtum, &c.

Note, "The Reaſon why I have reduced this Theorem to have the ſame Diviſor with thoſe at the Fruſtums of Pyramids, &c. will beſt appear farther on, viz. when they all come to be applied to practice in Gauging."

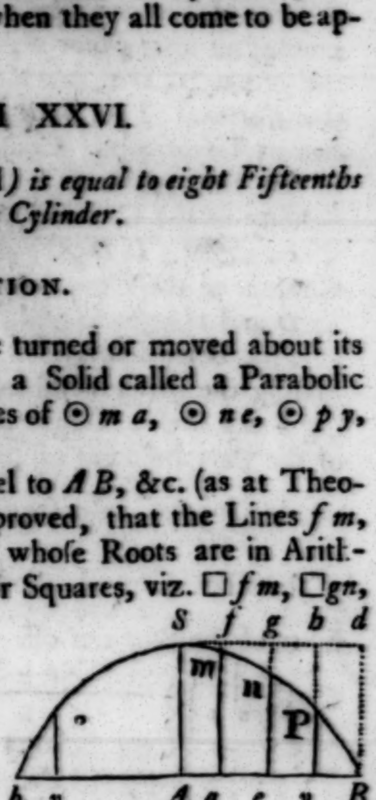
THEOREM XXVI.

*Every Parabolic Spindle (*or Pyramidoid*) is equal to eight Fifteenths of its circumſcribing Cylinder.*

Demonstration.

If any acute Parabola, as bSB, be turned or moved about its greateſt Ordinate bAB, it will form a Solid called a Parabolic Spindle, conſtituted of an infinite Series of $\odot ma$, $\odot ne$, $\odot py$, &c. by Definition 18.

Let us ſuppoſe the Line Sd, parallel to AB, &c. (as at Theorem 23) then it hath already been proved, that the Lines fm, gn, bp, &c. are a Series of Squares whoſe Roots are in Arithmetical Progreſſion: conſequently their Squares, viz. $\square fm$, $\square gn$, $\square bp$, &c. will be a Series of Biquadrates, whoſe Roots will be in Arithmetical Progreſſion: which being premiſed, we may proceed thus.



First, $\left\{ \begin{array}{l|l} 1 & SA - fm = ma \\ 2 & SA - gn = ne \\ 3 & SA - bp = py \end{array} \right.$

1 ⊙²

1 ●²	4	$\square SA - 2SA \times fm + \square fm = \square ma$
2 ●²	5	$\square SA - 2SA \times gn + \square gn = \square ne$
3 ●²	6	$\square SA - 2SA \times bp + \square bp = \square py$, &c.

In these Equations the $\square SA$, $\square SA$, $\square SA$ being a Series of Equals, and AB the Number of all the Terms; therefore it will be $\square SB \times AB =$ the Sum of the Series, by Lemma 1.

2. Because fm, gn, bp, &c. are as a Series of Squares wherein SA is the greatest Term, and AB the Number of all the Terms; therefore $\frac{2SA \times SA \times AB}{3} = \frac{2\square SA \times AB}{3}$ will be the Sum of all that Series, by Lemma 3.

3. And the $\square fm$, $\square gn$, $\square bp$, &c. will be a Series of Terms in the Ratio of Biquadrates, as above; $\square dB = \square SA$ being the greatest Term, and AB the Number of all the Terms; therefore it will be $\frac{\square SA \times AB}{5} =$ the Sum of all that Series, by Lemma 5.

Whence it follows, that $\square SA \times AB - \frac{2\square SA \times AB}{3} + \frac{\square SA \times AB}{5} =$ the Sum of all the Series of $\square ma$, $\square ne$, $\square py$, &c. That is, $\frac{8\square SA \times AB}{15} =$ the Sum of all the Series of $\square ma$, $\square ne$, $\square bp$, $\square dB$. &c. consequently, $\frac{8 \odot SA \times AB}{15} =$ the Sum of all the Series of $\odot ma$, $\odot ne$, $\odot py$, &c. which do constitute the Solidity of half the Spindle, viz. of SAB. Therefore putting $D = 2SA$, and $H = 2AB$, (viz. bAB) it will be $0,41888\ DDH =$ the Solidity of the whole Parabolic Spindle bSB, being $\frac{8}{15}$ of $0,7854\ DDH$ the Solidity of its circumscribing Cylinder. Q. E. D.

From hence we we may also raise a Theorem for finding the Frustum $SApy$ of the last Figure. For $\odot SA$ being the greatest Term, $\odot py$ the least Term, and Ay the Number of all the Terms or Circles included between A and y.

Therefore	1	$\left\{ \overline{\square SA - \frac{2SA \times bp}{3} + \frac{\square bp}{5}} \times Ay = z \right.$ the Sum of all the Series $\square SA$. $\square ma$, $\square gn$, $\square py$
1×3	2	$3\square SA - 2SA \times bp + \frac{3\square bp}{5} \times Ay = 3z$

2 ÷ Ay

$2 \div Ay$	3	$3\square SA - 2SA \times bp + \frac{3\square bp}{5} = \frac{3z}{Ay}$
But	4	$\square SA - 2SA \times bp = \square py - \square bp$, by 6th Step.
$3 - 4$	5	$2\square SA + \frac{3\square bp}{5} = \frac{3z}{Ay} - \square py + \square bp$
$5 +$ &c.	6	$2\square SA + \square py - \frac{2}{5}\square bp = \frac{3z}{Ay}$
Conseq.	7	$\overline{2\odot SA + \odot py - \frac{2}{5}\odot bp} \times \frac{1}{3}Ay = z$, the *Sum*

of all the Series of $\odot SA$, $\odot ma$, $\odot ne$, $\odot py$, which do constitute the Solidity of the Frustum $SApy$. Therefore putting $D = 2SA$, as before, $C = 2py$, $x = zbp$, and $H = Ay$, it will be $\overline{1,5708\,DD + 0,7854\,CC - 0,31416\,xx} \times \frac{1}{3}H =$ the Frustum $SApy$. And if we make $L = 2H$. Then $\overline{1,5708\,DD + 0,7854\,CC - 0,31416\,xx} \times \frac{1}{3}L =$ Double of that Frustum, being the middle Zone. And by turning these Factors into one common Divisor, as in the Frustum of the Conoid at Theorem 25, Page 430, there will arise this following Theorem.

THEOREM XXVII.

$$\left\{ \begin{array}{l} 3,8196)\ \overline{2DD + CC - 0,4xx} \times L\ (= \\ \text{the middle Zone of a Parabolic Spindle.} \end{array} \right.$$

It may be here expected that I should now proceed to shew how the Area of any Hyperbola, and the Contents of such Solids as may be formed by the Rotation of that Figure about its Axis, &c. may be found; but because those Things cannot be exactly performed by any certain or settled Theorem, as these of the Circle, Ellipsis, and Parabola have been, I have therefore omitted them, and Refer the Reader to Dr. *Wallis*'s Algebra, Chap. 90, &c. or to the Philosoph. Transact. Numb. 34, wherein he may find the Method of forming infinite Series relating to the squaring of an Hyperbola, &c. which are too tedious to be fully explained and demonstrated in this small Tract, it being only intended as an Introduction, the which I shall here conclude.

FINIS.

AN APPENDIX OF PRACTICAL GAUGING.

THE Art of Gauging is that Branch of the Mathematics called *Stereometry*, or the meaſuring of Solids, becauſe the Capacities or Contents of all Sorts of Veſſels uſed for Liquors, &c. are computed as tho' they were really ſolid Bodies; which any one that hath made himſelf Maſter of the foregoing Parts of this Treatiſe may eaſily underſtand, without any farther Directions.

However, becauſe it is not to be ſuppoſed that every one, who deſigns to undertake the Office or Employment of a Gauger, hath made ſo great a Progreſs in Mathematical Learning, I have therefore preſented the young Gauger with this Appendix, wherein I have only inſerted ſuch Rules as are uſeful in Gauging, and have been already demonſtrated in this Treatiſe. But herein, I preſuppoſe that he hath acquired (or if not, it is very neceſſary he ſhould acquire) a competent Knowledge both in Arithmetic and Geometry: That is,

I. In Arithmetic he ſhould underſtand the principal Rules very well, eſpecially Multiplication and Diviſion, both in whole Numbers and Decimal Parts, (which may be eaſily learnt out of the 2d, 3d, and 5th Chapters of Part 1.) that ſo he may be ready at computing the Contents of any Veſſel, and caſting up his Gauges by the Pen only, *viz.* without the Help of thoſe Lines of Numbers upon ſliding Rules, ſo much applauded, and but too much practiſed, which at beſt do but help to gueſs at the Truth; I mean ſuch Pocket-Rules as are but nine Inches (or a Foot) long, whoſe Radius of the double Line of Numbers is not ſix Inches; and therefore the Graduations or Diviſions of thoſe Lines are ſo very cloſe, that they cannot be well diſtinguiſhed. It is true, when the Rules are made two or three Feet long (I had one of ſix Feet) there they may be of ſome Uſe, eſpecially in ſmall Numbers; altho' even then the Operations may be much better (and almoſt as ſoon) done by the Pen: For, indeed, the chief Uſe of Sliding-Rules is only in taking of Dimenſions, and for that Purpoſe they are very convenient.

II. In Geometry the Gauger ſhould underſtand not only how to take Dimenſions (which is beſt learnt by Practice) but alſo how to divide any irregular Figure or Superficies, as Brewers Backs or Coolers, &c. into the eaſieſt and feweſt regular Figures they will admit of, that ſo their Areas may be truly computed with the leaſt Trouble. And this may be learned (with a little Care and Diligence) out of the 1ſt, 2d, and 5th Chapters of Part III, which the Gauger ſhould be well acquainted with. Alſo he ought to have ſo much Skill in Solids, as to be able, even at Sight (but this muſt be acquired by Experience) to determine what ſort of Figure any Veſſel is of (viz. any Tun or cloſe Caſk) or what Figures it may be beſt reduced to, ſo that its Dimenſions may be truly taken, and the Content thereof computed with the leaſt Error. I ſay, with the leaſt Error, becauſe it is very difficult, if not impoſſible, to do it exactly; for there is not any Tun, or cloſe Caſk, &c. ſo regularly made, as by the Rules of Art it is required to be.

III. Beſides the aforementioned, the young Gauger muſt know, that all Dimenſions uſeful in Gauging are to be taken in Inches, and Decimal Parts of an Inch; and if they are taken in any other Meaſures, as Feet, Yards, &c. thoſe Meaſures muſt be reduced to Inches, (ſee Sect. 4. Pag. 42.) becauſe the Contents of all Sorts of Veſſels (taken Notice of in Gauging) are computed by the Standard Gallon of its Kind, whoſe Content is known to be a certain Number of Cubic Inches: That is, the Beer or Ale Gallon contains 282, the Wine 231, and the Corn Gallon 268, eight Cubic Inches. [See the five Tables, &c. in Pages 34, 35, 36, which I here ſuppoſe the Gauger to have learnt perfectly by heart.] Conſequently, if either the ſuperficial or ſolid Content of any Veſſel, as Back, Tun, Caſk, &c. be once computed in Cubic Inches, it will be eaſy to know how many Gallons, either of Ale, Wine, or Corn, that Veſſel will hold.

Note, I have here ſaid, the ſuperficial Content in Cubic Inches, which may ſeem to be very improper, according to the Definition given of a Superficies in Page 285; but you muſt know, that, in the Buſineſs of Gauging, all Superficies or Areas are always underſtood to be one Inch deep, otherwiſe it could not be ſaid (as in the Gauger's Language it is) that the Area of ſuch a Back, or of ſuch a Circle, &c. is ſo many Gallons.

Theſe Things being very well underſtood, the young Gauger will be fitly prepared to underſtand the following *Problems*, which are ſuch as have (moſt of them) been already propoſed in the foregoing Parts of this Treatiſe, and only are here applied to Practice; and therefore I ſhall, for Brevity Sake, often refer to thoſe Theorems and Problems.

Sect.

Sect. 1. To find the *Area* of any right-lined Superficies in Gallons.

PROBLEM I.

To find the Area *of any* square Tun, Back, *or* Cooler, *&c. either in* Ale, Wine, *or* Corn Gallons.

Rule. Multiply the given Length or Breadth (being here equal) into itself, and the Product will be the Area in Inches; then divide that Area by 282, or 231, or 268,8 and the Quotient will be the Area required.

Example. Suppose the Side of a Square Tun, Back, or Cooler be 124,5 Inches, what will its Area be in Gallons?

First $124,5 \times 124,5 = 15500,25$ the Area in Inches.

Then 282	15500,25	54,96 &c.	the *Area* in	*Ale Gallons.*
And 231		76,10 &c.		*Wine Gallons.*
Or 268,8		57,66 &c.		*Corn Gallons.*

But if any one would rather work by Multiplication than by Division, he may turn or change any Divisor into a Multiplicator, if he divide Unity, or 1, by that Divisor. (Vide Probl. 3. Page 408.)

Thus 282	1,000000	0,003546	the Multip. for	*Ale Gallons.*
And 231		0,004329		*W. Gallons.*
Or 268,8		0,003722		*C. Gallons.*

Consequently $15500,25 \times 0,003546 = 24,96$ &c. the Area in Ale Gallons, as before; and so on for the rest.

PROBLEM II.

To find the Area *of any* Tun, Back, *or* Cooler *in the Form of a Right-angled Parallelogram in* Ale Gallons, *&c.*

See the Rule for finding its Area in Inches, at Probl. 1. P. 339, then either divide (or multiply) that Area, as above, and you will have the Area in Gallons.

Example, Suppose the Length of a Brewer's Tun, Back, or Cooler be 217,5 Inches, and its Breadth 85,6 Inches, what will its Area be in Ale or Beer Gallons, *&c?*

First $217,5 \times 85,6 = 18648$. Then 282) 18648 (66,12, &c.
Or $18648 \times 0,003546 = 66,12$, &c. the *Area* required, &c.

PROBLEM III.

To find the Area *of any* Triangular Tun, Back, *or* Cooler, *in* Ale Gallons, *&c.*

See the Rule for finding its Area in Inches at Prob. 3, p. 340: then divide (or multiply) that Area as before, and you will have the Area required.

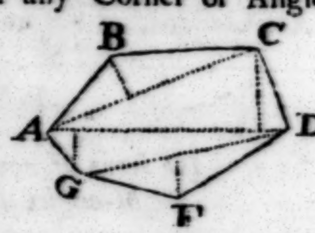
Example. If the Length of the Base of a Triangular Cooler be 86,4 Inches, and its perpendicular Breadth be 57 Inches, what will its Area be in Ale Gallons?

First, $86{,}4 \times \frac{67}{2} = 2462{,}4$. Then 282) 2462,4 (8,73 *&c.* Or $2462{,}4 + 0{,}003546 = 8{,}73$ *&c.* the Area in Ale Gallons.

Proceeding thus, you may easily find the Area of any Tun, Back, or Cooler, whether it be in the Form of a Rhombus, Rhomboides, Trapezium, or of any other Polygon, either regular or irregular, in Ale or Beer Gallons, *&c.* if you first divide it into Triangles, and then find the Areas of those Triangles; (as in the 2d, 4th, 5th, and 6th Problems in Chap. 5, Part III.) the Sum of those Areas being divided (or multiplied) by its proper Divisor (or Multiplicator) as above, will give the Area required.

Now, the practical Way of dividing any Polygonous Tun, Back, &c. into Triangles, is by help of a chalked Line, such as the Carpenters use, and may be thus performed.

Suppose any Brewer's Tun, Back, or Cooler in the Form of the annexed Figure *ABCDFG*. Let one End of the chalked Line be fastened with a Nail (or otherwise) in any Corner or Angle of the Back, as at *A*; then straining it to the Angle at *C*, strike the Diagonal Line *AC* upon the Bottom of the Back; and straining it again to the Angle *D*, strike another Diagonal Line, as *A D*, and so on for the Diagonal Line *G D*, &c. Then having



marked out all the Diagonals, the Perpendiculars may be thus found: Fasten (as before) one End of the chalked Line, in the Angle *B*, and then, by moving it *to* and *fro* upon the Stretch, find out the nearest Distance between the Angle at *B* and the Diagonal Line *AC*; and there strike a Line, and it will mark out the Perpendicular from *B* to the Line *AC*, and so on for the other Perpendiculars: Which being all marked out upon the Bottom of the Back, measure them and each Diagonal by a Line of Inches,

Inches, &c. and then the Area of that Back may be computed, as directed above.

And here, by the Way, it may be obſerved, that the Number of Triangles will always be leſs by two, and the Number of the Diagonals leſs by three, than the Number of the Sides of any Right-lined Figure that is ſo divided.

Having found (as above) the true Area of any Brewer's Back or Cooler (which, according to the Laws of Exciſe, ought always to be fixed or immoveable) the next Thing will be to find out the true dipping or gauging Place in that Back, ſo that the true Quantity of Worts may be computed or (caſt up) at any Depth; which may be thus done.

1. When the Bottom of the Back is covered all over (of any Depth) either with Worts or Liquor (viz. *Water)* then dip it in eight or ten ſeveral Places (more or leſs according to the Largeneſs of the Back) as remote and equally diſtant one from another as you well can, noting down the wet Inches, and decimal Parts of every Dip.

2. Divide the Sum of all thoſe Dips or wet Inches by the Number of Places you dipped in, and the Quotient will be the mean Wet of all thoſe Dips.

3. Laſtly, find out ſuch a Place by the Side of the Back (if you can) that juſt wets the ſame with that mean Dip, and make a Notch or Mark there, for the true and conſtant Dipping-place of that Back. Then if any Quantity of Worts (which do cover the whole Back) be dipped or guaged at that Place, and the wet Inches ſo taken be multiplied into the Area of the Back in Gallons, the Product will ſhew what Quantity (viz. how many Gallons) of Worts are in that Back at that Time, provided the Sides of the Back do ſtand at Right Angles with its Bottom.

Sect. 2. *To find the* Area *of any* Circular *and* Elliptical Superficies *in Gallons.*

1. I have demonſtrated in Chap. 6. Part III, and Theorem 3. 5, 6, Part V. that the Periphery of the Circle whoſe Diameter is Unity, or 1, is 3,14159265 &c. (or for common Uſe 3,1416) and that its Area is 0,78539816 &c. (or 0,7854 *fere.)*

2. Alſo, that the Peripheries of all Circles are in Proportion one to another as their Diameters are; and their Areas are in Proportion to the Squares of the Diameters. That is, as 1 : 3,1416 : : the Diameter of any Circle : to its Periphery. And 1 : 0,7854 : : the Square of the Diameter : to the Area.

Upon

Upon these two Proportions depend the Solutions of all the common or practical Questions about a Circle. [See Page 408, 409.]

PROBLEM IV.

The Diameter *of any* Circle *being given in Inches, to find the* Periphery.

RULE. { Multiply the given Diameter with 3,1416, and the Product will be the Periphery required. [See Prob. 1. p. 408.]

Example. Suppose the Diameter of a Circle be 54,5 Inches, and it were required to find its Periphery. Then $54,5 \times 3,1416 = 171,21.$ &c. Inches in the Periphery required. The Converse of this is easy, *viz.* by having the Periphery given, to find the Diameter. [See Prob. 3. Page 408.]

PROBLEM V.

The Diameter *of any* Circle *being given* (in Inches) *to find its* Area *in Gallons.*

RULE. { Multiply the Square of the proposed Diameter into 0,7854, and the Product will be the Area in Inches; [See Probl. 2. P. 408.] that Area being divided by 282, or 231, &c. the Quotient will be the Area required.

Example. Suppose the given Diameter be 54,5 Inches as above. First, $54,5 \times 54,5 = 2970,25$. And $2970,25 \times 0,7854 = 2332,83$ the Area in Inches:

$$\left.\begin{array}{l}\text{Then } 282\\ \text{And } 231\\ \text{Or } 268,8\end{array}\right\} 2332,83 \left\{\begin{array}{r}8,2724\\ 10,0988\\ 8,6788\end{array}\right\} \text{the Area in} \left\{\begin{array}{l}\text{Ale or Beer Gals.}\\ \text{Wine Gallons.}\\ \text{Corn Gallons.}\end{array}\right.$$

But these Areas in Gallons may be much easier found without knowing the Circle's Area in Inches, as above, by having the Square of the Diameter of that Circle whose Area is one Gallon; which may be thus found, by Theorem 6, Page 407.
0,785398 : 1 :: 282 : 359,05 the Square of the Diameter of the Circle whose Area is 282 cubic Inches, *viz.* one Ale Gallon.

And from this Proportion will arise the following Divisors;

$$viz.\ 0,785398 \left\{\begin{array}{l}282,000000(359,05\\ 231,000000(294,12\\ 268,800000(342,24\end{array}\right\} \text{will be a } Divis. \text{ for} \left\{\begin{array}{l}A.\ G.\\ W.\ G.\\ C.\ G.\end{array}\right.$$

If

If the Square of the Diameter of any Circle be divided by any one of these constant or fixed Divisors, the Quotient will shew that Circle's Area in their respective Gallons. As for Instance, in the last Circle, whose Square of its Diameter is 2970,25.

$$\left.\begin{array}{l}\text{Then } 359{,}05\\ \text{And } \ \ 294{,}12\\ \text{Or } \ \ \ 342{,}24\end{array}\right\}\ 2970{,}25\ \left\{\begin{array}{r}8{,}2725\\ 16{,}0988\\ 8{,}6788\end{array}\right\}\text{the } \textit{Area} \text{ in}\left\{\begin{array}{l}\textit{A. G.}\\ \textit{W. G.}\\ \textit{C. G.}\end{array}\right\}\text{as bef.}$$

Now these Divisors may be turned into Multiplicators by dividing Unity or 1, as in Page 433: Or rather by dividing the Area in Inches of that Circle whose Diameter is 1.

That is, 0,785398 by 282. Or by 231, &c.

$$\left.\begin{array}{l}\text{Thus } 282\\ \text{And } \ \ 231\\ \text{Or } \ \ 261{,}8\end{array}\right\}\ 0{,}785398\ \left\{\begin{array}{r}0{,}002785\\ 0{,}003399\\ 0{,}002922\end{array}\right\}\text{the } \textit{Multiplic.} \text{ for}\left\{\begin{array}{l}\textit{Ale} \text{ Gal.}\\ \textit{W.} \text{ Gal.}\\ \textit{C.} \text{ Gal.}\end{array}\right.$$

These Multiplicators are the respective Areas of a Circle whose Diameter is 1; and therefore, if the Square of the Diameter of any Circle be multipled with any of these Numbers, the Product will be that Circle's Area in Gallons of the same Name:

Vid. 2970,25 × 0,002785 = 8,2725 the Area in *A.G.* as above.
And 2970,25 × 0,003399 = 10.0988 the Area in *W. Gal.* &c.

Thus you see, that, if the Diameter of any Circle be given in Inches, there are three several Ways of finding its Area in Gallons, and all equally true; but that which is performed by the constant Divisors is most generally practised.

PROBLEM VI.

The Transverse (or longest Diameter) *and the Conjugate* (or shortest Diameter) *of any* Elliptical Superficies *being given, to find its* Area *in Gallons.*

RULE. { Multiply the two Diameters (*viz.* the Length and Breadth) together, and divide their Product by 359,05 for Ale Gallons, or 294, 12 for Wine Gallons, &c. the Quotient will be the Area required. [See Theorem 7, Page 412.]

Example. Suppose the longest Diameter to be 73,5 Inches and the shortest Diameter to be 51,6 Inches; what will the *Area* be in Ale Gallons.

First 73,5 × 51,6 = 3792,6. Then 359,05) 3792,6 (10,56 the Area in Ale Gallons. Or 294,12) 3792,6 (12,89 the Area in Wine Gallons, &c.

Note,

Note, The two last Problems are of great Use in Gauging of Worts amongst Country Victuallers, who generally brew but short Lengths of Ale (perhaps between 20 and 60 Gallons at a Brewing) and cool their Worts in several small open Vessels or Tubs, whose Bases or Bottoms are either a Circle, or an Ellipsis, having their Sides but low, and are most commonly wider at the Top than at the Bottom.

Now a practical Way of computing the Quantity of Worts, that are at any Time in one of those open Tubs, is briefly thus: When the Tub is dry, find the true Area of its Bottom according to its Figure (as above) and either mark that Area on the Outside of the Tub (which was the Way I generally used to order, because the Victuallers did often lend their cooling Tubs one to another) or else number the Tub, and enter its Area (and its Number) into the Stock-book; then, when any of those Tubs hath Worts in it, take the Diameter of the Surface or Top of the Worts, and find that Area, adding it and the bottom Area together. If either the half Sum of those two Areas be multiplied with the Depth of the Worts (taken as near the Middle of the Tub as you well can) or, if the Sum of those two Areas be multiplied with half the Depth (so taken) the Product will shew the Quantity of those Worts very near the Truth.

PROBLEM VII.

The Diameter *of any* Circle, *and the versed Sine*, viz. (the Height) *of any Segment, being given, to find the* Area *of that Segment in* Gallons.

In the 410th and 412th Pages you have two Ways (and their Examples) of finding the Area of any Segment of a Circle in Inches; then if that Area in Inches be divided by 282, or 231, *&c.* the Quotient will be its Area in Gallons. But because the Area of any such Segment may be readily found in Gallons (without finding its Area in Inches) by Help of a Table of Segments, whose Construction is laid down in the Problem, Page 411, &c. I have here inserted a Compendium of such a Table, which will serve very well for common Practice, not only to find the Area of any Segment of a Circle in Gallons, but also to find the Number of Gallons, that are either drawn out, or remaining in any Cylindric Vessel lying along, or of any close Cask (being first reduced to a Cylinder) its Axis lying parallel to the Horizon, usually called the *Ullage* of a Cask; as shall be shewed farther on.

A Table

A Table of the Segments of a Circle whose Area is Unity or 1, the Diameter being divided by parallel Chord-Lines into 100 equal Parts.

V. S.	Segment	V. S.	Segment.	V. S.	Segment.	V. S.	Segment.
1	0,0017	26	0,2066	51	0,5127	76	0,8155
2	0,0048	27	0,2178	52	0,5255	77	0,8262
3	0,0087	28	0,2292	53	0,5382	78	0,8369
4	0,0134	29	0,2407	54	0,5509	79	0,8474
5	0,0187	30	0,2523	55	0,5635	80	0,8576
6	0,0245	31	0,2640	56	0,5762	81	0,8677
7	0,0308	32	0,2759	57	0,5888	82	0,8776
8	0,0375	33	0,2878	58	0,6014	83	0,8873
9	0,0446	34	0,2998	59	0,6140	84	0,8968
10	0,0520	35	0,3119	60	0,6265	85	0,9059
11	0,0598	36	0,3241	61	0,6389	86	0,9149
12	0,0680	37	0,3364	62	0,6514	87	0,9236
13	0,0764	38	0,3486	63	0,6636	88	0,9320
14	0,0851	39	0,3611	64	0,6759	89	0,9402
15	0,0941	40	0,3735	65	0,6881	90	0,9480
16	0,1032	41	0,3860	66	0,70[illegible]2	91	0,9554
17	0,1127	42	0,3986	67	0,7122	92	0,9625
18	0,1224	43	0,4112	68	0,7241	93	0,9692
19	0,1323	44	0,4238	69	0,7360	94	0,9755
20	0,1424	45	0,4365	70	0,7477	95	0,9813
21	0,1526	46	0,4491	71	0,7593	96	0,9866
22	0,1631	47	0,4618	72	0,7708	97	0,9913
23	0,1738	48	0,4745	73	0,7822	98	0,9952
24	0,1845	49	0,4873	74	0,7934	99	0,9983
25	0,1955	50	0,5000	75	0,8045	100	1,0000

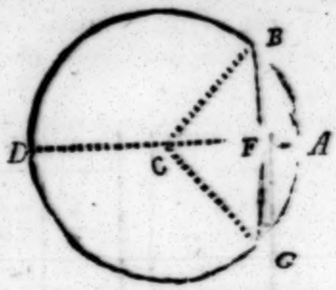
The Use of this Table of Segments depends upon the following Proportion:

viz. As the Diameter of any proposed Circle : is to 100 (the Diameter of the tabular Circle) : : so is the Height of any Segment of the proposed Circle : to a versed Sine in the Table.

Then, if the tabular Segment, which stands against that versed Sine, be multiplied into the Circle's Area (either in Inches or Gallons) the Product will be the Area of the Segment required [of the same Name] *viz.* If the Circle's Area be Inches, the Segment will be Inches; if Gallons, the Segment will be Gallons.

 Example,

Example. Let the Diameter of the given Circle be $DA=62,5$ Inches, and the Height of the Segment sought be $FA=20$ Inches; What will its Area be in Ale Gallons?



First, the Area of the whole Circle will be 10,8793 Ale Gallons (by Problem 5.) and the Proportion will stand thus, 62,5 : 100 : : 20 : 32 the versed Sine of the Table whose Segment is 0,2759. Then, $10,8720\times0,2759=$ 3,0016 Ale Gallons, being the Area of the Segment *BAGF*, as was required. The like may be done for Wine Gallons, Corn Gallons, or Inches.

And, upon Occasion, the like Segments of any Ellipsis may be easily found. See the *Proportions* in the Corollaries to the 7th and 8th Theorems, Page 412, *&c.* to which I here, for Brevity's Sake, refer the Reader.

Sect. 3. To compute the Contents of such Vessels, (viz. Tuns, &c.) as are in the Form of the following Solids.

Note, Before the young Gauger proceeds to these Computations, he should be well acquainted with such Solids as are defined in P. 402 and 403, and then he may easily understand what Sort of Figures are meant in the following Problems, without the Repetition of many Words.

PROBLEM VIII.

To find the Content of any Prism *whose Sides are* Parallelograms *what Form soever its Base is of.*

That is, to compute the Content (in Gallons) of any Tun, &c. whose Sides are Parallelograms which stand upright, or at Right Angles with its Bottom.

First, find its solid Content in Inches, by Theorem 9, Page 414; then divide that Content by 282, or 231, or by 268,8; the Quotient will shew the Content in their respective Gallons, *viz.* in Ale, Wine, or Corn Gallons.

Or else multiply the Content in Inches with 0,003546, or 0,004329, &c. [See the Multiplicators, Page 435] those Products will be the Content in their respective Gallons.

Or otherwise thus:

Find the true Area of the Tun's Base or Bottom, as directed in Sect. 1. P. 435; that Area being multiplied with the Tun's Height (*viz.* Depth within) will produce the Content in Gallons, as before.

I take

I take the Work of this Problem to be so very easy, that it needs no Example.

PROBLEM IX.

To find the Content of any Pyramid (in Gallons) *whose Base is bounded with Right Lines.*

Every Pyramid is one third Part of its circumscribing Prism, by Theorem 10, Page 415. Therefore, if the Area of the Base of any Pyramid, in Gallons, be multiplied into one Third of its perpendicular Height; or if one Third of that Area be multiplied with the whole Height, either of those Products will be the Content of the Pyramid in Gallons, &c. But the Content of any square Pyramid may be easily found in Gallons by this Rule:

RULE. Square the Side of its Base, and multiply that Square with the perpendicular Height; then divide that Product by $846=282\times 3$ for Ale Gallons, or by $693=231\times 3$ for Wine Gallons, or by $806,4=268,8\times 3$ for Corn Gallons, the Quotient will be the Content required.

Or, if you multiply the said Product with 0,001182 for *A. G.* or with 0,001443 for *W. G.* or, lastly, with 0,001241 for *C. G.* the Result will be the Content required, as before.

PROBLEM X.

To find the Content (in Gallons) *of the* Frustum *of any square Pyramid, cut off by a Plane parallel to its Base.*

First, Either by Theorem 15, Page 419, or Theorem 16, P. 420, find the proposed Frustum's Solidity in Cubic Inches; then divide that Content in Cubic Inches by 282 or 231, &c. and the Quotient will be the Content of the Frustum in their respective Gallons.

But, from the foresaid Theorem 15, there may be easily deduced the following general Rule for finding the Content of the like Frustum of any Pyramid, what Form soever its Bases are of (supposing them to be parallel) whether they are alike or unlike.

RULE. First, find the Area of each Base, (*viz.* the top and bottom Areas of the proposed Frustum;) then find a Geometrical Mean between those two Areas (by Lemma 1, Page 77;) the Sum of those two Areas and their Mean, being multiplied into one Third of the Frustum's Height, will produce the Content required.

Example. Suppose a Tun in the Form of the lower Frustum of a Pyramid, whose Bases are equilateral Triangles: Let the Side of the Top be 42 Inches, the Side of the Bottom be 63,4 Inches, and its Height [*viz.* Depth] be 33 Inches; What will the Content of that Tun be in Ale Gallons?

First, find the Area of each Base in Inches, by Probl. 7, P. 343; then find what those Areas are in Ale Gallons, by Probl. 3. P. 436. Multiply those two Areas together and the square Root of their Product will be the mean Area, &c. as in this Example:

Example. The $\left\{\begin{matrix}\text{Top}\\ \text{Bottom}\\ \text{Mean}\end{matrix}\right\}$ Area is $\left\{\begin{matrix}2{,}71\\ 6{,}12\\ 4{,}07\end{matrix}\right\}$ Ale Gallons.

Their Sum 12,90

Then $12{,}9 \times \frac{23}{3} = 141{,}9$. Or $\frac{12{,}9}{3} \times 33 = 141{,}9$ the Content required.

PROBLEM XI.

To find the Content of any right Cylinder in Gallons.

That is, to compute the Content of any round Tun, &c. whose Diameters at Top and Bottom are equal, and at Right-angles with its Sides.

The Content of such a Tun may be found by Theorem 11, Page 415; or otherwise by the following Rule.

RULE. Multiply the Square of the Diameter into the Height, and divide the Product by 359,05 (or multiply with 0,002785) &c. as in Page 439, that Quotient (or Product) will be the Content required.

Example. Suppose the Diameter be 42,5, and the Height 31,5 Inches.

First $42{,}5 \times 42{,}5 = 1806{,}25$. And $1806{,}25 \times 31{,}5 = 56896{,}875$.
Then 359,05) 56896,875 (158, 46 the Content in Ale Gal. &c.

PROBLEM XII.

To find the Content of any Cone *or* round Pyramid *in Gallons.*

Because every Cone is one Third of its circumscribing Cylinder, [See Theorem 13, Page 416] therefore its Content may be truly found by the following Rule.

RULE. Multiply the Square of the Diameter of its Base into the perpendicular Height, then divide their Product by $1077{,}15 = 359{,}05 \times 3$ for Ale Gallons, or by $882{,}36 = 294{,}12 \times 3$ for Wine Gallons, &c. and the Quotient will be the Content required.

Or

Or if the said Product be multiplied with 0,000928 $= \frac{0,002785}{3}$, or with 0,001133 $= \frac{0,0034}{3}$, those Products will be the Content in their respective Gallons.

Example. Suppose the Diameter of the Base be 42,5, and the perpendicular Height be 31,5 Inches, What will the Content be in Ale Gallons? (as before.

First 42,5×42,5=1806,25. And 1806,25×31,5=56896,875
Then 1077,15) 56896,875) 52,82. Or 56896,25×0,000928 =52,82 the Content in Ale Gallons. And so on for Wine or Corn Gallons.

PROBLEM XIII.

To find the Content of the lower Frustum *of any Cone in Gallons.*

That is, to compute the Content of any round Tun, &c. whose Diameters at Top and Bottom are parallel, but unequal.

The Content of such a Tun may be found by the Rule at Problem 10; but from Theorem 16, Page 420, it will be easy to deduce this following Rule.

RULE. To the triple Product of the top and bottom Diameters, add the Square of their Difference; multiply that Sum into the Height, (or Depth): then divide the last Product by 1077,15 for Ale Gallons, or by 882,36 for Wine Gallons; the Quotient will be the Content required.

Example, Suppose the Diameter at the Top to be 52,4 Inches, the Diameter at the Bottom 44,6, and the Height 30 Inches.

1st, 52,4×44,6=2337,04; and 2337,04×3=7011,12 } Add
Also, 52,4—44,6=7,8; and 7,8 × 7,8= 60,84 }
The Height 30× 7071,96=212158,8

Then 1077,15(212158,8 (196,96 } the *Content* in *Ale* Gallons.
Or 212158,8×0,000928=196,96 }

And so on for either Wine or Corn Gallons, as Occasion requires. But if the Tun (or Vessel) be not truly circular, that is, if either its Top or Bottom (or both of them) be elliptical, whether they are alike or unlike, it matters not, the Content of such a Tun may be truly found by the general Rule at Problem 10.

PROBLEM XIV.

The Axis *or* Diameter *of any Sphere or Globe being given in* Inches *to find its Content in* Gallons.

Every Sphere is two Thirds of its circumscribing Cylinder, by Theo. 18, Page 423; from whence and Theor. 20, P. 426, it is proved,

proved, that if the Cube of the Axis of any Sphere (taken in Inches) be multiplied into 0,5236, the Product will be the Content of that Sphere in Inches. Consequently, if that Content be divided by 282, or by 231, &c. the Quotient will be the Content in Gallons.

But those two Works of multiplying with 0,5236, and then dividing by 282, or by 231, &c. may be contracted into one.

$$\begin{array}{l}\text{Thus } 282\\ \text{And } 231\end{array}\Big\}\ 0{,}5236\ \Big\{\begin{array}{l}0{,}001856\\ 0{,}002266\end{array}\Big\}\text{ will be a } \textit{Multiplicat.} \text{ for } \Big\{\begin{array}{l}\textit{A. G.}\\ \textit{W.G.}\end{array}$$

$$\text{Or } 0{,}5236\Big\}\begin{array}{l}282\\ 231\end{array}\Big\{\begin{array}{l}538{,}57\\ 441{,}17\end{array}\Big\}\text{ will be a } \textit{Divisor} \text{ for } \Big\{\begin{array}{l}\textit{Ale}\text{ Gallons.}\\ \textit{W.}\text{ Gallons.}\end{array}$$

From hence arises this following Rule.

RULE. If the Cube of the Axis of any Sphere be divided by 538,57; or multiplied with 0,001856: or divided by 441,17; or else multiplied with 0,002266; the Quotient, or Product, will be the Sphere's Content in their respective Gallons.

Example. Suppose the Axis or Diameter of a Sphere or Globe be 22 Inches, how many Ale Gallons may it hold?

Here $22\times22\times22=10648$; and 538,57) 10648 (19,76 *A. G.* Or $10648\times0{,}001856=19{,}76$ Ale Gal. the Content required. And so for either Wine or Corn Gallons, as Occasion requires.

PROBLEM XV.

To find the Content of a Segment *of a* Sphere *in Gallons.*

In the Scholium, P. 424. there are two Theorems for resolving this Problem according to the Data.

1. If the Diameter of the Segment's Base and its Height are given, the Content may be found by the first of those Theorems, which gives this Rule:

RULE 1. To the triple Square of half the Diameter add the Square of the Height; then multiply that Sum into the Height, and divide the Product by 538,57 for *A. G.* or by 441,17 for *W. G*, &c. as above.

2. But if the Axis of the Sphere and the Height of the Segment are given, the Content may be found by the second of those Theorems.

RULE 2. From the triple Product of the Axis into the Height, subtract twice the Square of the Height; then multiply the Remainder into the Height, and divide that Product by 538,57, &c. as in the last Problem.

Either

Either of these Rules will produce the Content of the Segment in Gallons.

Example. Suppose the Diameter of the Segment's Base be 28 Inches, and its Height be 8 Inches, what may it contain in Ale Gallons?

First 2) 28 (14. Then (by Rule 1.) $14 \times 14 \times 3 = 588$. And $6 \times 6 = 36$. Next $588 + 36 = 624$. Again $624 \times 6 = 3744$. Lastly, 538,57) 3744 (6,95 the Content required.

Note. This Problem may be of Use in Gauging the Crowns of Brewer's Coppers, &c.

Sect. 4. *The practical Method of Gauging any fixed Tun or Copper, and making a Table to shew what it will hold at every Inch deep, usually called Inching of a Tun,* &c.

FIRST, you must know, that most (if not all) Brewer's Tuns are so fixed as to lean a little for Conveniency of cleansing their Drink, which is usually called the Drip or Fall of the Tun. Now this Drip or Fall of any Tun is the Hoof of such a Solid as that Tun is supposed to represent, and under that Consideration it may be found, as in Theor. 16, P. 420: But the practical (and indeed the best) Way is, to measure into the Tun (when it is dry) so much Liquor as will just cover its Bottom; for by that means you do not only find the true Fall, but also a true horizontal or level Plane over the Bottom of the Tun; from which if the Depth of the Tun (*viz.* the nearest Distance from the Top of the Tun to the Surface of the Liquor) be set off upon every one of its Sides, you will then have a true parallel Plane at the Top of the Tun to that of the Liquor. Then, if the Sides of the Tun are streight from the Top to the Bottom, take as many Dimensions in the aforesaid two Planes as are needful to find the true Area of each; and by those two Areas and the aforesaid Depth find so much of the Tun's Content (by the general Rule at Problem X.) as is betwixt those two Planes.

Next, to inch that Tun, divide the Difference between the Top and Bottom Areas by the aforesaid Depth, and the Quotient will be an Addend or fixed Number; which being added to the lesser Area, the Sum will be the Area of the next Inch; and, being added to that Area, their Sum will be the Area of the third Inch; and so on from Inch to Inch, until the Area of every single Inch be found; the Sum of those Areas (if the Work be true) will amount (or be equal) to the Content found, as above. And if the

the Tun's Drip or Fall be added to the Sum of all thoſe Areas, that Sum will be the whole or full Content of that Tun.

Now, from hence it muſt needs be eaſy to conceive, that if 1. 2, 3, or any Number of thoſe Areas accounted from the Bottom, be added to the Fall, that Sum will ſhew the Quantity of Liquor or Drink that is in the Tun, to ſuch a Number of wet Inches from the Bottom as there were Areas added together. Or, if the Sum of any Number of thoſe Areas (being accounted from the Top) be ſubtracted from the Tun's whole Content, the Remainder will ſhew what Quantity of Liquor or Drink is in the Tun, when there is ſuch a Number of dry Inches from the Top as there were Areas ſubtracted.

This being well conſider'd, it will be eaſy to make a Table either to every wet or dry Inch of any regular Tun (*viz.* whoſe Sides are ſtreight from Top to Bottom) what Form ſoever its Baſes are of, and whether it ſtand upon the greater or leſſer Baſe.

But if the Sides of the Tun are irregular (*viz.* not ſtreight from its Top to the Bottom) then the beſt and eaſieſt Way will be to divide or part the Tun into ſeveral Fruſtums, each of ten Inches deep; and finding the Content of every ſingle Fruſtum, by taking the Diameters in the Middle of every one of thoſe ten Inches (that is, the firſt Diameter at 5 Inches from the Top; the ſecond Diameter at 15 Inches from the Top, &c.) and multiplying their reſpective Areas with 10, (which is done only by removing the ſeparate Comma's one Place forward to the right Hand) if the Sum of all thoſe Fruſtums be added to the Fall (as before); that Sum will be the whole Content of the Tun.

Note, If you take the Height of the 'foreſaid ten Inch Fruſtums in the Side of the Tun, you muſt allow for the Difference between the ſlant Height and the perpendicular Height in every Fruſtum.

Laſtly, If from the whole Content of the Tun you ſubtract the mean Area of the firſt Fruſtum ten Times, and from the Remainder ſubtract the mean Area of the ſecond Fruſtum ten Times, and from the laſt Remainder ſubtract the mean Area of the third Fruſtum, &c. until there remain nothing but the Fall or Hoof of the Tun, you will then by that Means have a Table that will ſhew what Quantity of Drink is in the Tun to any Number of dry Inches.

And this is alſo the Method of Gauging and Inching Brewer's Coppers, *viz.* by firſt meaſuring into the Copper ſo much Liquor as will juſt cover its Crown, and then dividing its perpendicular Height into Fruſtums, and its Sides into four equal Parts; that ſo croſs Diameters may be taken in the Middle of each Fruſtum: But

but if the Copper be much wider at the Top than at the Bottom, and its Sides ſpheroidal or arching, as generally all large Coppers are; then, inſtead of taking thoſe mean Diameters in the Middle of every ten Inches, as above, you muſt take them in the Middle of every ſix Inches, and proceed on as before.

Now the Quantity of Liquor, that would cover the Crown of the Copper, may be found without meaſuring it, as above. In order to that, I do ſuppoſe the Crown to be the Segment of a Sphere, and the lower Part of the Copper wherein the Crown ariſeth, to be the Fruſtum of a parabolic Conoid; then, if the Diameter at the Top of the Crown, and its perpendicular Height are given, the Quantity of Liquor may be found by this following Rule:

RULE. { From the Area of the Plane at the Top of the Crown ſubtract $1\frac{1}{3}$ of the Area of the Crown's Height; the Remainder, being multiplied into half the Height of the Crown, will produce the Quantity or Number of Gallons that will cover the Crown.

This *Rule* is deduced from *Scholium*, Page 424, and *Theorem* 25. Page 430.

Sect. 5. *To compute the Content of any cloſe Caſk in* Gallons, *viz.* *of any* Butt, Pipe, Hogſhead, Barrel, &c.

IN order to perform this difficult Part of Gauging, the three following Dimenſions of the propoſed Caſk muſt be truly taken in Inches, and Decimal Parts of an Inch.

Viz. { The Bulge or Bung Diameter within the Caſk.
Either of the Head Diameters, ſuppoſing them both equal.
And the Length of the Caſk within.

Note, In taking of theſe Dimenſions, it muſt be carefully obſerved,

1. That the Bung-hole be in the Middle of the Caſk; alſo that the Bung-ſtaff and the Staff over-againſt the Bung-hole are both regular or even within.

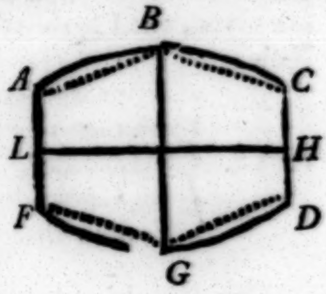
2. That the Heads of the Caſk are equal and truly circular; if ſo, the Diſtance between the Inſide of the Chine to the Outſide of its oppoſite Staff will be the Head Diameter within the Caſk, very near.

3. With a ſliding Pair of Calipers (made on Purpoſe for that Uſe) take the ſhorteſt Diſtance or Length between the Outſides of the two Heads; (ſuppoſing them even) from that Length ſubtract $1\frac{1}{2}$ Inch (more or leſs, according to the Largeneſs of the Caſk)

Cask) for the Thickness of the two Heads, the Remainder will be the Length of the Cask within.

Now, by these Dimensions, one would suppose the Content of the Cask were perfectly limited; but it will be easy to perceive, by the following Figure, that the Diameters (abovesaid) and the Length of one Cask may be equal to those of another, and yet one of those Casks may contain or hold several Gallons more than the other.

As for Instance, suppose the annexed Figure *ABCDGF*, to represent a Cask; then it is plain, that, if the outward and curved Lines *ABC* and *FGD* are the Bounds or Staves of the Cask, it must needs hold more than if the inner streight or pricked Lines were its Bounds or Staves; and yet the Bung Diameter *BG*, Head Diameter *CD* and *AF*, and the Length *LH*, are the same in both those Casks.



Whence it plainly appears, that no one certain or general Rule can be prescribed to find the true Content of all Sorts of Casks, and therefore Gaugers do usually suppose every Cask to be in Form of some one of these following Solids.

Viz.
- I. The middle Zone or Frustum of a Spheroid.
- II. The middle Zone or Frustum of a Parabolic Spindle.
- III. The lower Frustums of two equal Parabolic Conoids.
- IV. The lower Frustums of two equal Cones.

Now the Way of Guessing at the Cask's Form, and computing its Content, according to its supposed Form, I shall here shew in their Order.

I. If the Staves of the Cask are very much curved or arching (as the outward Lines of the last Figure) then the Cask is supposed to be in the Form of the middle Zone or Frustum of a Spheroid, whose Content may be computed, by *Theorem* 22, *Page* 427, which gives these two Rules.

RULE I. To twice the Square of the Bung Diameter add the Square of the Head Diameter; multiply that Sum into the Length, and divide the Product by 1077,15. *Viz.* 3,8197×282 for Ale Gallons; and by 882,36. *Viz.* 3,8197×231 for Wine Gallons. Or thus,

RULE

Rule 2. To twice the Area of the Bung Circle, add the Area of the Head Circle; multiply their Sum into one Third of the Length, and the Product will be the Content in their respective Gallons.

Example 1. Suppose a Cask in the Form of the middle Zone of a Spheroid, whose Bung Diameter is 31,5, Head Diameter 24,3, and its Length 42 Inches.

First $31,5 \times 31,5 \times 2 = 1984,5$. And $24,5 \times 24,5 = 600,25$
Again $1984,5 + 600,25 = 2584,74$. And $2584,75 \times 42 = 108559,5$
Then 1077,15) 108559,5 (100,78 the Content in Ale Gallons.
And 882,35) 108559,5 (123,03 the Content in Wine Gallons.

Or thus, by the Second Rule.

Bung Diameter 31,5 twice its Circle's Area is 5,5270
Head Diameter 24,5 its Circle's Area is 1,6718
The Length 42 divided by 3 is 14. 7,1988 = their Sum.
Then $7,1988 \times 14 = 100,78$, the Content in A. Gallons as before.
And so the Content in Wine Gallons may be found.

II. If the Staves of the Cask are not quite so much curved or arching, as was supposed before, the Cask is then taken for the middle Frustum of a parabolic Spindle, and its Content is computed, as by Theorem 27, Page 432. Which gives this Rule.

Rule. To twice the Square of the Bung Diameter add the Square of the Head Diameter; from their Difference subtract 4 Tenths of the Square of the Difference of the Diameters; multiply the Remainder into the Length, and divide the Product by 1077,15, &c. as above.

Example 2. Suppose the Dimensions the same as before. Then $31,5 \times 31,5 \times 2 : + 24,5 \times 24,5 = 2584,75$. And $31,5 - 24,5 = 7$. Again $7 \times 7 \times 0,4 = 19,6$. And $2584,75 - 19,6 \times 42 = 107736,3$. Then 1077,15) 107736,3 (100,01 the Cont. in *A. G.* &c. for *W. G.*

III. When the Staves of the Cask are but very little curved or arching, then it is supposed to be in the Form of the Frustums of two equal parabolic Conoids, abutting or joining together upon one common Base at the Bulge, and the Content may be found by Theorem 25, Page 430. which gives these Rules.

Rule 1. To the Square of the Bung Diameter add the Square of the Head Diameter; multiply their Sum into the Length, and divide the Product by 718,08 (*viz.* $2,5464 \times 282$) for Ale Gallons: or by 588,22 (*viz.* $2,5464 \times 231$) for Wine Gallons. Or thus,

RULE 2.	To the Area of the Bung Circle add the Area of the Head Circle; multiply the Sum into half the Length; and the Product will be the Content required.

Example 3. With the same Dimensions as before. Then
$315 \times 31,5 + 24,5 \times 24,5 = 1592,5$. And $1592,5 \times 42 = 66885$
And 718,08) 66885 (93,01 the Content in Ale Gallons.
Or 588,22) 66885 (113,7 the Content in Wine Gallons.

IV. If the Staves of the Cask are streight from the Bulge to the Head, as the inner prick'd Lines in the last Figure (if such a Cask can be made) it is then taken for the lower Frustums of two equal Cones, abutting or joining together upon one common Base at the Bulge. And its Content may be computed as at Problem 13, *Page* 445, or by Theorem 15, *Page* 415. Thus,

RULE.	To the Sum of the Squares of the Head and Bung Diameters add their Product; then multiply that Sum into the Length, and divide the last Product by 1077,15. Or by 882,36. The Quotient will be the Content, &c.

Example 4. With the same Dimensions as before.
First $31,5 \times 31,5 + 24,5 \times 24,5 + 31,5 \times 24,5 = 2364,25$
And $2364,25 \times 42 = 99298,5$. Then 1077,15) 99298,5 (92,18 the Content in Ale Gallons, and so on for Wine Gallons.

Thus you have the Methods of computing the true Contents of the four Solids, in whose Form all Casks are supposed to be. And by the Examples it appears, that four such Casks as have their Dimensions all equal, and the same with those above-mentioned, their Contents will be as in the Margin.

	Ale Gallons	Differ.
I.	100,78	
II.	100,01	0,77
III.	93,01	7,00
IV.	92,18	0,83

From the Disproportion or Inequality of these Differences it will be easy to conceive, that there may be several Casks whose Contents cannot be truly found, according to the aforesaid supposed Forms; and therefore, in order to rectify the said Inequalities, some Authors (that have written upon this Subject) have laid down *Theorems* of their own Invention; (and yet called them by these Names) others have proposed Tables for the same Purpose. But since it is so, that we can only guess at the Truth, the plainest and easiest Way is to be preferred in Practice; and that is, by finding such a mean Diameter as will reduce the proposed Cask to a Cylinder.

Thus,

Thus, { Multiply the Difference between the Head and Bung Diameters, with 0,7. or with 0,65. or with 0,6. or with 0,55, according as the Staves of the Cask are more or less arching; add the Product to the Head Diameter, and the Sum will be the mean Diameter required. Then find the Content, as at *Prob.* 11. *Page* 444.

Example. With the same Dimensions as before. Then the Bung Diameter less the Head Diam. is 31,5—24,5=7. And

		M D.	*A G.*	*Cont.*	*Dif.*
25,5+	7×0,7	=29,40	its *Area* 2,4073×42=	101,10	
	7×0,65	=29,05	——— 2,3504×42=	98,71	2,39
	7×0,6	=28,70	——— 2,2941×42=	96,35	2,36
	7×0,55	=28,35	——— 2,2385×42=	94,03	2,32

From these it may be observed, that the Difference between each Cask's Content is regular, and very near equal; which plainly shews, that there is not so much Room left for Error this Way of computing their Contents, as was by the aforesaid Forms.

Now the first of these four (*viz.* with 0,7) is very commonly used among *Gaugers* for all Sorts of Casks; but I did never gauge any Cask that would contain quite so much as that Rule did make it; and the Reason doth appear very plain from *Theorem* 22, *Page* 427, being compared with *Theorem* 19, *Page* 426. and the last Figure, *viz.* that no Cask (being regularly made) can hold more than the middle Frustum of a Spheroid. But I always found by Experience, that if the second and third of these Rules (*viz.* with 0,65 and 0,6) were duly applied, they would answer very near the Truth amongst the common Sort of Casks; and the fourth Rule (*viz.* with 0,55) will come pretty near the Truth in computing the Contents of Casks, whose Staves are almost streight betwixt the Head and Bung, *viz.* such as Wine Pipes, &c.

Sect. 6. *To find what Quantity of Liquor is either drawn forth, or remaining in any spheroidal Cask, usually called the Ullage of a Cask; hath two Cases.*

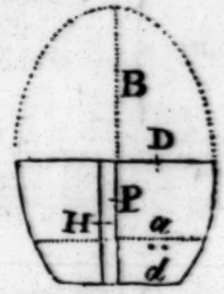
Case 1. TO find what Quantity of Liquor is in the Cask, when its Axis is perpendicular to the Horizon, *viz.* when it stands upright upon one of its Heads.

In order to perform this the easiest Way, it will be convenient to know how to calculate the Area of any Circle betwixt the Bung and Head, whose Distance from the Bung or Middle of the Cask is given. Now that may be done by this Proportion.

Viz. { As the Square of half the Length of the Cask : is to the Difference between the Bung and Head Areas : : so is the Square of any Circle's Distance from the Bung : to the Difference between the Bung Area, and the Area of that Circle, *viz.* the Area of the Liquor's Surface.

DEMON-

Demonstration.



Let $\begin{cases} H = \text{Half the Length of the Cask.} \\ D = \text{Half the Bung Diameter.} \\ d = \text{Half the Head Diameter.} \end{cases}$

And $\begin{cases} P = \text{the Distance of any Circle from the Bung} \\ a = \text{Half the Diameter of that Circle.} \end{cases}$

Then according to the common Property of the Ellipsis, Page 368, it will be,

$BB : DD :: BB - HH : dd.$ And $BB : DD :: BB - PP : aa,$

Ergo $\left\{ \dfrac{DDHH}{DD - dd} = BB. \right.$ And $\left\{ \dfrac{DDPP}{DD - aa} = BB. \right.$

Consequently, $\left\{ \dfrac{DDHH}{DD - dd} = \dfrac{DD - aa}{DDPP}. \right.$

This Equation, being brought out of the Fractions, will become $DDHH - aaHH = DDPP - ddPP$, which gives this *Analogy* $HH : DD - dd :: PP : DD - aa$. Then $DD - aa$ being subtracted from DD, will leave aa. But Circle's Areas are in Proportion to the Squares of their Diameters, by Theorem 6, Page 407. Therefore, &c. Q. E. D. Then, from the Bung Area subtract one third Part of the aforesaid Difference, *viz.* between the Bung Area and the Area of the Liquor's Surface; multiply the Remainder with the Liquor's Distance from the Bung, and the Product will shew what Quantity of Liquor is either above or under half the Content of the Cask.

Example. Let us suppose a Cask of the same Dimensions with that in the first *Example*, Page 451. and let it be required to find what Quantity of Liquor is in it (of Ale Measure) when there is but 9 Inches wet. Here half the Length of the Casks is 21 Inches, whose Square is 441, and the Liquor's Distance from the Bung is 21=9=12. Its Square is 144. The Difference between the Bung and Head Areas is 1,0917 (=2,7635 — 1,6718.) Then 441 : 1,0917 : : 144 : 0,3564.

And 2,7635 — 0,3564 = 2,4071 the Area of the Liquor's Surface.

Again 3) 0,3564 (0,1188. And 2,7635 — 0,1188 = 2,6447 Then 2,6447×12=31,7364, what the Cask wants of being half full. Consequently 50,39—31,73=18,66 will be the Quantity of Liquor in the Cask at 9 Inches wet in Ale Gallons.

And

And if the Cask had wanted but 9 Inches of being full; then $50{,}39+31{,}73=82{,}12$ would have been the Quantity of Liquor in the Cask.

Note, because the two first Terms (*viz.* 441 and 1,0917) in the Proportion are fixed, *viz.* continue the same for any Distance, it will be very easy to calculate the Areas of all the Circles betwixt the Bung and Head to every Inch, and by that Means to make a Table that will shew what Quantity of Liquor is either drawn out, or remaining in the Cask at any Depth.

Case 2. To find what Quantity of Liquor is in any Cask, when its Axis is parallel to the Horizon, *viz.* when it lies along.

There are Variety of Tables to be found in Books of Gauging for this Purpose; but I always observed, that the following Method of computing the Ullage, by a Table of the Segments of a Circle, came very near the Truth in all Sorts of Casks, which is thus perform'd:

1. By the Bung and Head Diameters, find such a mean Diameter as you judge will reduce the propos'd Cask to a Cylinder, by the Method laid down in *Page* 453. And then find its full Content, as in those *Examples*.

2. From the Bung Diameter subtract the mean Diameter, and half the Difference, (*viz.* divide it by 2.)

3. From the wet Inches of the propos'd Ullage, subtract the said half Difference, and call it x; then observe this Proportion.

Viz. { As the mean Diameter : is to 100 (the Diameter of the tabular Circle) : : so is the last Difference (*viz.* x) : to a versed Sine in the Table. (Page 441.)

Then if the tabular Segment, which stands against that versed Sine, be multiply'd into the Content of the Cask, the Product will shew the Ullage, *viz.* what Quantity of Liquor is either in the Cask, or drawn forth.

Example 1. Let the Cask be that of the second Sort, in *Page* 453, *viz.* whose Bung Diameter is 31,5 Inches, mean Diameter 29,05, and the Content 98,71 Ale Gallons; and suppose there were 10,5 Inches wet in it, it is required to find the wet and dry Gallons?

Here $31{,}5-29{,}05=2{,}45$; its half is 1,22. And $10{,}5-1{,}22=9{,}28$
Then $29{,}05 : 100 :: 9{,}28 : 31{,}9=$ V. Sine; its Segm. is 0,2748
And $98{,}71\times0{,}2748=27{,}12$ the Number of wet Gallons.

Again

Again 31,5—10,5=21 the dry Inches; and 21—1,22=19,78
Then 29,05 : 100 :: 19,78 : 68,0; its Segment is 0,7241
And 98,71×0,7241=71,48 the Number of dry Gallons.
Proof 71,48+27,12=98,6 the Contents of the Cask very near; which plainly shews the Truth of this Method.

Thus far may suffice concerning Gauging of Backs or Coolers, Tuns, Coppers, and Casks, &c. To which I shall only add, that as the Contents of all Brewers Utensils are to be computed by the Ale Gallons, so the Contents of all Distillers Utensils (viz. all their Wash-backs, Stills, and Casks, &c.) must be computed by the Wine Gallon.

And in gauging of Malt (upon which there is now a Duty of four Shillings per Bushel) you must observe, That a Corn or Malt-Bushel doth contain 2150,42 cubic Inches; (See Page 36.) and therefore in gauging of Malt-Cisterns, or other Vessels 2150,42 will be a constant or fixed Divisor for finding the Areas of right-lined Figures in Bushels at one Inch deep, and 2738 will be a constant or fixed Divisor for finding the Areas of circular Figures.

I have omitted the Business of gauging Mash-Tuns, and taking an Account of the Goods or Grains, in order to estimate what Quantity of Worts were produced from them, &c. because I could never find (by all my Observations) any Certainty therein; nor is it possible there should be any, by Reason of the great Difference that is in Malt (and its Grinding too) for the best Malt (well ground) will yield or produce the most Worts, and least Grains; on the contrary, bad Malt (being ill ground) yields the least Worts and most Grains.

A SUP-

A

SUPPLEMENT

Not in the former EDITIONS of this

BOOK.

Containing the

HISTORY

OF

LOGARITHMS,

WITH

Several easy METHODS of Constructing the TABLES of the LOGARITHMS and SINES, &c. Also the Demonstration of the AXIOMS and Doctrine of PLANE

TRIGONOMETRY.

Extracted from the

PHILOSOPHICAL TRANSACTIONS and the WORKS of Dr. KEIL, RONAYNE, WARD, &c.

Cuncta Trigonus *habet, satagit quæ docta* Mathesis,
Ille *aperit clausum quicquid* Olympus *habet.*

PREFACE.

THE Mathematics formerly received considerable Advantages; first, by the Introduction of the Indian *Characters, and afterwards by the Invention of Decimal Fractions; yet has it since reaped at least as much from the Invention of Logarithms, as from both the other two. The Use of these, every one knows, is of the greatest Extent, and runs through all Parts of Mathematics. By their Means it is that Numbers almost infinite, and such as are otherwise impracticable, are managed with Ease and Expedition. By their Assistance the Mariner steers his Vessel, the Geometrician investigates the Nature of the higher Curves, the Astronomer determines the Places of the Stars, the Philosopher accounts for other Phænomena of Nature; and lastly, the Usurer computes the Interest of his Money.*

The Subject of the following Treatise has been cultivated by Mathematicians of the first Rank; some of whom, taking in the whole Doctrine, have indeed wrote learnedly, but scarcely intelligible to any but Masters. Others, again, accommodating themselves to the Apprehension of Novices, have selected out some of the most easy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouched. My Design therefore, in the following Tract, is to supply what seemed still wanting, viz. *to discover and explain the Doctrine of Logarithms, to those who are not yet got beyond the Elements of Algebra and Geometry.*

The wonderful Invention of Logarithms we owe to the Lord Neper, *who was the first that constructed and published a Canon thereof, at* Edinburgh, *in the Year* 1614. *This was very graciously received by all Mathematicians, who were immediately sensible of the extreme Usefulness thereof. And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet* Neper *is universally allowed the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.*

The same Lord Neper *afterwards invented another and more commodious Form of Logarithms, which he afterwards communicated to Mr.* Henry Briggs, *Professor of Geometry at* Oxford, *who was hereby introduced as a Sharer in the com-*

pleting thereof: But, the Lord Neper *dying, the whole Business remaining was devolved upon Mr.* Briggs, *who, with prodigious Application, and an uncommon Dexterity, compassed a Logarithmic Canon, agreeable to that new Form for the first twenty Chiliads of Numbers (or from* 1 *to* 20000*) and for* 11 *other Chiliads,* viz. *from* 90000 *to* 1010000. *For all which Numbers he calculated the Logarithms to* 14 *Places of Figures. This Canon was published at* London *in the Year* 1624.

Adrian Vlacq *published again this Canon at* Goudæ *in* Holland *in the Year* 1628, *with the intermediate Chiliads before omitted, filled up according to* Briggs's *Prescriptions; but these Tables are not so useful as* Briggs's, *because the Logarithms are continued but to* 10 *Places of Figures.*

Mr. Briggs *also has calculated the Logarithms of the Sines and Tangents of every Degree, and the hundredth Parts of Degrees to* 15 *Places of Figures, and has subjoined to them the natural Sines, Tangents, and Secants, to* 15 *Places of Figures. The Logarithms of the Sines and Tangents are called artificial Sines and Tangents, but the Sines and Tangents themselves are called natural. These Tables, together with their Construction and Use, were published after* Briggs's *Death, at* London, *in the Year* 1633, *by* Henry Gellibrand, *and by him called* Trigonometria Britannica.

Since then, there have been published, in several Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, consist of but seven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 *to* 100000, *which may be sufficient for most Uses.*

The best Disposition of these Tables, in my Opinion, is that, first thought of by Nathaniel Roe, *of* Suffolk; *and, with some Alterations for the better, followed by* Sherwin *in his Mathematical Tables published at* London *in* 1705; *wherein are the Logarithms from* 1 *to* 101000 *consisting of* 7 *Places of Figures. To which are subjoined the Differences and proportional Parts, by means of which may be found easily the Logarithms of Numbers to* 10000000, *observing at the same Time that these Logarithms consist only of* 7 *Places of Figures. Here are also the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with some other Tables of Use in practical Mathematics.*

THE

THE CONSTRUCTION OF LOGARITHMS.

THESE most excellent and useful Numbers were first invented by the famous and never to be forgotten Lord *Neper*, Baron of *Merchiston* in *Scotland*, aforesaid (*Ann.* 1614.) who ingeniously contrived to perform Multiplication and Division of natural Numbers, by only adding or subtracting certain artificial Numbers, which he called *Logarithms*, and the Extraction of Roots by dividing the Log. by 2 for the Square: by 3 for the Cube: by 4 for the Biquadrate, &c.

This Invention of his (no doubt) proceeded from a mature Consideration of the Coherence that is betwixt Numbers in Geometrical Proportion and those in Arithmetical Progression.

As in these following:

Viz. $\begin{cases} 1 \,.\, 2 \,.\, 4 \,.\, 8 \,.\, 16 \,.\, 32 \,.\, 64 \,.\, 128, \text{\&c. Geometrical.} \\ 0 \,.\, 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7, \text{\&c. Arithmetical.} \end{cases}$

It is very perceptible, that, as the Numbers in the Geometrical Proportionals are produced by *Multiplication* or *Division*, those in the Arithmetical Progression are produced by *Addition* or *Subtraction*: As doth appear in this Example:

Viz. $\begin{Bmatrix} 4 \times 32 = 128 \\ 2 + 5 = 7 \end{Bmatrix}$ or $\begin{cases} 128 \div 32 = 4 \text{ Geometr.} \\ 7 - 5 = 2 \text{ Arithmet.} \end{cases}$

Again, $\begin{cases} 1 \,.\, 10 \,.\, 100 \,.\, 1000 \,.\, 10000 \,.\, 100000, \text{\&c. Geometr.} \\ 0 \,.\, 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5, \text{\&c. Arithmet.} \end{cases}$

The same Coherence is betwixt these latter, as was between the two first Ranks.

Viz. $\begin{Bmatrix} 1000 \times 10 = 10000 \\ 3 + 1 = 4 \end{Bmatrix}$ or $\begin{cases} 100000 \div 1000 = 100 \text{ Geometr.} \\ 5 - 3 = 2 \text{ Arithmet.} \end{cases}$

Either of these Examples do sufficiently shew the Reason and very Ground of Logarithms.

And from the latter of these it was, that the prime Logarithms or Characteristics were first assigned.

As in this Table:

Natural Num.	*Logarithms.*
1	0,0000000
10	1,0000000
100	2,0000000
1000	3,0000000
10000	4,0000000
100000	5,0000000

Having laid this Foundation, the next Work was to find out the Logarithms of the intermediate Numbers situated betwixt 1 and 10, *viz.* 2, 3, 4, 5, 6, 7, &c. and of those betwixt 10 and 100, *viz.* 11, 12, 13, 14, 15, &c. and so on for the rest. This was a Work of some Difficulty, and very laborious.

The first Step in order thereunto (as I conceive) was to find out a Rank of continual Means betwixt 10 and 1, so as that the last (and least thereof) might be a mixed Number less than 2, and so near 1, as to have such a Number of Cyphers before the significant Figures thereof, as was intended the Places of Logarithms in the Table should consist of. Which Means are to be found, by extracting the square Root of 10 (having first annexed a competent Number of Cyphers thereunto;) then extracting the Root of that Root, and so by a continued Extraction of Root out of Root, until there be a Root so qualified as before-mentioned: Which, to make a Table to seven Places in the Logarithms, will require twenty-five several Extractions, the last of which will produce this Number, 1,0000000686223S.

The next Step was to find out a Number betwixt [1] and [0] in Arithmetical Progression, that might truly correspond with the Means before found [betwixt 10 and 1] such a Number must consequently be its Logarithm. And this may be found by a continual bisecting or [halving] of 1, so often as was the Number of the foregoing Extractions [to wit, twenty-five] the last of which Bisections will produce 0,0000000298023 22, &c. the true Logarithm of 1,00000006862238.

For as 1,00000006862238 by twenty-five continued Involutions [*viz.* first into itself, then that Product into itself, and so on successively] will produce 10; so will 0,0000000298023 2, by the the like Number of Doublings and Redoublings, produce 1.

This Mean [or Number] and its Logarithm being thus found, it will follow by Proportion, *As the significant Figures of this Mean : are to the significant Figures of its Logarithms : : so are the significant Figures of any Mean, betwixt any given Number and* 1 :

[having

[having seven Cyphers before such Figures, as this hath] *to the significant Figures of its Logarithm.* To which must be prefixed seven Cyphers to complete it. After which, being doubled, and redoubled according to the Number of Extractions required to produce its corresponding Mean, will at last discover the true Logarithm of the given Number. For the clearing of this, take an Example.

Suppose it were required to find the Logarithm of the Number 2, to seven Places. First, by a continued Extraction of Root out of Root, beginning at 2, find such a Mean, or Root as before, betwixt 2 and 1, as will have seven Cyphers before its significant Figures; which after twenty-five several Extractions, will be this Number 1,00000008262958. Then, according to the foregoing Proportions, it will be 6862238 : 2980232 : : 8262958 : 3588557. To which prefix seven Cyphers, as before directed, then will 1,00000008269958 have for its Logarithm, ,00000003588557; which being doubled and redoubled, as abovesaid, will produce 0,30102997958658, the true Logarithm of 2; which being contracted to seven Places, according to the first Design, and agreeable to the seven places of Cyphers, then it will become 0,3010299. But, in all the Tables that I have seen, the Logarithm of 2 is 0,3010300: I conceive the Reason is, because the remaining Figures 7958658 come so near Unity of the last Place in the retained Figures.

And, by the same Method that this Logarithm of 2 is made, may the Logarithm of any other Number be found. But when once the Logarithms of a few of the prime Numbers, *viz.* of 3. 7. 11. 13, &c. [that is, of such Numbers as cannot be produced by the multiplying of two Integer Factors] are obtained, the rest may be easily composed by *Addition* and *Subtraction* only. For as $3 \times 2 = 6$ so Log. of 3 + Log. of 2 = Log. of 6. And as $10 \div 2 = 5$ so Log. of 10 — Log. of 2 = Log. of 5. The like of all Numbers that have aliquot Parts (that is, such Integer Numbers as may be divided by Integers.) And indeed the Logarithms of several of the prime Numbers may also be obtained by *Addition* or *Subtraction*, as might easily be shewed, and is not difficult to conceive by any one, who but duly considers the Nature and Design of Logarithms, &c, of which I shall forbear saying any Thing in this Place, and keep to my first Design herein, which was to give a brief Account of the ingenious Author's Method, as I conceive it, of making the same: who undoubtedly found it a very difficult Work, by Reason there are required so many several Extractions of Roots out of Roots, which must needs render it both troublesome and laborious. Then to propose a different Method of raising the Logarithms of such prime Numbers before-mentioned, which require the Extraction of Roots to obtain their respective Means, with one

one tenth Part of the Trouble and Time required by the foregoing Method. And not only so, but more exact; for, by our present Method of converging Series, the Root of any Power, how high soever it be, is easily found at one single Extraction; and thereby the Errors which would arise by extracting a *Surd Root* out of a *Surd Root*, especially when often repeated, are avoided; and consequently such a Mean, as may be required betwixt any Number and Unity, is thereby more exactly found.

Now, how this may be performed, I here intend to shew, as briefly as I can. In order thereunto, I take this as a Model.

Let $a =$ the Root, or Mean required betwixt any Number and Unity:

$$\text{Then}\begin{cases} a^2 = \square a \quad . \; a^4 = \square a^2 \quad . \; a^8 = \square a^4 \\ a^{16} = \square a^8 \quad . \; a^{32} = \square a^{16} \quad . \; a^{64} = \square a^{32} \\ a^{128} = \square a^{64} \; . \; a^{256} = \square a^{128} \; . \; a^{512} = \square a^{256} \end{cases}$$

And so on successively with the Indices in Geometrical Progression, until the Power of a be made equal to such a Term in that Progression, as that the Root, or Value of a may have, betwixt Unity and its significant Figures, so many Cyphers, as are the intended Number of Places in the Logarithms.

For Instance, let it be required to find the Mean between 10 and 1, then the Power of a must be $a^{33554432} = 10$, this Index 33554432 being the 25th Term in Geometrical Progression, which may be thus determined.

Let 1, the Characteristic or Logarithm of 10, be divided by such a Term in Geometrical Progression, as will cause such a Number of Cyphers to be before the significant Figures in the Quotient, as are required to be before the Figures of the Root a; suppose 7, as before. Then 1, ÷ 33554432 = ,0000000298023 2, &c. which is the true Arithmetical Mean (as before found, by a continual bisecting of 1) correspondent to that signified by a; and therefore the Value of a found by extracting the respective Root of $10 = a^{33554432}$ will be the Mean required; *viz.* 1,0000000686223 8 whose Log. is ,0000000298023 2. These, being found, are the Foundation of the rest, as before.

Then suppose it be required to find the Logarithm of any of the prime Numbers; if you please, that of two. In order thereunto, let $a =$ the Root or Mean sought betwixt 2 and 1, as before; then must a be continually involved, as by the above Model, until its Index be equal to the greatest Term in Geometrical Progression, whose Number of Places of Figures are to be equal to the Number of required Cyphers before a, to wit 7. According to which, the Power of a will be $a^{8388608} = 2$ (this 8388608 being the 23d Term in Geometrical Progression) consequently the respective Root of $2 = a^{8388608}$ will be the Mean required.

Example.

Example.

Let $r+e=a$

Then will $r^{8388608}+8388608\,r^{8388607}e$
$+35184367894528\,r^{8388606}ee=a^{8388606}=2$

Suppose $r=1$

Then $1+8388608e+35184367894528ee=2$

That is $8388608e+35184367894528ee=1$

Each Part being divided by the Co-efficient found prefixed to ee, viz. 351843, &c. then it will become

$$,00000023e+ee=,000000000000284=D$$

$$\text{Consequently }\left\{\frac{,00000023+e}{D}=e\right.$$

$$,000000000000284=D$$

,00000023	248	(,00000008$=e$
$+e=$,00000008	36	
Divisor ,00000031		

First $r=1$,
$+e=$,00000008

New $r=$1,00000008

which being duly involved, in the same Order as the Model denotes, and multiplied into the respective Co-efficients, will then produce these Numbers,

Viz. $1{,}956368967+16411168e+68833416066289ee=2$

Then $16411168e+68833416066289ee=,0436361033$

And $,0000002384e+ee=0000000000000063393=D$

$$\text{Consequently }\left\{\frac{D}{,0000002384+e}=e\right.$$

$$,0000000000000063393=D$$

,0000002384	480	(,0000000263$=e$
$+e=$,0000000026		
Divisor ,000000240	15393	
	14400	
Divisor ,0000002410	9330	
	7230	

Last $r=$ 1,00000008
$+e=$,0000000263

New $r=$1,0000008263

I take only 1,0000000268 $= r$; the which being involved, and ordered as before, will produce these following Numbers, *viz.*

$1{,}999503684867 + 16773028e + 70351267454084ee = 2$

Then $16773028e + 70351267454084ee = {,}000496315133$

$,0000002384186e + ee = 00000000000000000705481443 = D$

Consequently $\left\{ \dfrac{D}{,0000002384186 + e} = e \right.$

	,00000000000000000705481443 $= D$
,0000002384186	47686 (,000000000295 $= e$
$+ e =$,000000000295	
	2286214
Divisor ,00000023843	2146023
Divisor ,000000238447	14019143
	11922405
Divisor ,0000002384481 *	
	* 20967380 (879 $= e$
Here I desist forming a new Divisor, and make use of the Abridgment.	19075849
	1891532
	1669136
	222396
Last $r =$ 1,0000000826	214596
$+ e =$,0000000000295879	
$a =$ 1,0000000826295879	

This Value of $a =$ 1,0000000826295879 is the Geometrical Mean betwixt 2 and 1, as was required; [agreeable to that before found, by twenty-three several Extractions.) And by this Method of proceeding, may be found the Mean betwixt 10 and 1, *viz.* 1,0000006862238, or betwixt any other of the (before-mentioned) Prime Numbers and Unity, as might easily be shewed. But for Brevity Sake, I shall omit giving more Examples thereof, this one being sufficient (especially to the Ingenious) if well considered, and but once understood, to shew the Nature of, and Manner how to proceed on the like Occasion, of finding any proposed Mean. The next Thing will be to find the Logarithm of the Number from whence such Mean was produced, which may be thus performed.

First, find its corresponding Arithmetical Mean, or Logarithm, by Proportion (as in Pag. 462.) then multiply that corresponding Mean, so found, into the Index Number of such Power as the Geometrical Mean was produced from; that Product will be the Logarithm

Logarithm of the given Number (without a continued Doubling and Redoubling, as before.) For the clearing of this, let it be required to complete the Logarithm of 2.

Having first found 1,00000006862238, the proper Geometrical Mean betwixt 10 and 1; also its corresponding Logarithm ,00000002980232 (as before directed) with them and the Mean betwixt 2 and 1, last found, *viz.* 1,000000008262958 79; make use of the above-mentioned Proportion (as in Pag. 463.) *viz.*

6862238 : 2980232 :: 826295879 : 358855729

To which prefix seven Cyphers to complete it (as before.) Then it will become ,0000000358855729. This Number being multiplied into the Power of *a* (what that is, see Page 465) will produce the Logarithm of 2.

viz. 0000000358855729×8388608=0,30103000391352

But according to the first Design, it is required to have but seven Places, *viz.* 0,301300; which is the true Logarithm of 2 without any Defect.

Thus I have presented you with a new and expeditious Method of making Logarithms; which if they were required to fourteen or fifteen Places (I can modestly say) they might then be made with one twentieth Part of the Time and Trouble required by the first Method.

METHOD III.

A New Table of Logarithms. *Composed by Mr.* LONG. *Finding the* Logarithm *by* Division *only, and the Natural Number belonging to a* Logarithm, *by* Multiplication *only.*

Log.	*Nat. Num.*	*Log.*	*Nat. Num.*
0,9	7.943282347	0,00009	1.000207254
0,8	6.309573445	0,00008	1.000184224
0,7	5.011872336	0,00007	1.000161194
0,6	3.981071706	0,00006	1.000138165
0,5	3.162277660	0,00005	1.000115136
0,4	2.511880432	0,00004	1.000092106
0,3	1.995262315	0,00003	1.000069080
0,2	1.584893193	0.00002	1.000046053
0,1	1.258915412	0,00001	1.000023026
0,09	1.230268771	0,000009	1.000120724
0,08	1.202264435	0,000008	1.000018421
0,07	1.174897555	0,000007	1,000016118
0,06	1.148153621	0,000906	1.000013816
0,05	1.122018454	0,000005	1.000011513
0,04	1,096478196	0.000004	1,000009210
0,03	1.071519305	0,000003	1.000006908
0,02	1.047128548	0,000002	1.000004605
0,01	1.023292992	0,000001	1.000002302
0,009	1.020939484	0,0000009	1.000002072
0 008	1.018591388	0,0000008	1.000031842
0,007	1.016248694	0,0000007	1.000001611
0,006	1.013911386	0,0003006	1.000001381
0,005	1.011579454	0,0000005	1 000001151
0,004	1.009252886	0,0000004	1.000000921
0,003	1.006931669	0,0000003	1.000000690
0,002	1.004615794	0,0000002	1.000000460
0,001	1.002305238	0,0000001	1.000000230
0,0009	1.002074475	0,00000009	1.000000107
0,0008	1.001843766	0,00000008	1.000000184
0,0007	1.001613106	0,00000007	1.000000161
0,0006	1.001382509	0,00000006	1.000000138
0,0005	1.001151956	0,00000005	1.000000115
0,0004	1.000921459	0,00000004	1.000000092
0,0003	1.000691015	0,00000003	1.000000069
0,0002	1.000460623	0,00000002	1.000000046
0.0001	1.000230285	0,00000001	1.000000023

This

This Table I sometimes make use of for finding the Logarithm of any Number proposed, and *vice versa*. Suppose I had Occasion to find the Logarithm of 2000. I look in the first Class of my Table (the whole table consists of 8 Classes) for the next less to 2, which is 1.995262315, and against it is 3, which consequently is the first Figure of the Logarithm sought. Again, dividing the Number proposed 2, by 1.995262315 the Number found in the Table, the Quotient is 1.002374467; which being looked for in the second Class of the Table, and finding neither its Equal, nor a Lesser, I add 0 to the Part of the Logarithm before found, and look for the said Quotient, 1.002374467 in the third Class, where the next less is 1.002305238, and against it is 1, to be added to the Part of the Logarithm already found; and dividing the Quotient 1.002374467, by 1.002305238, last found in the Table, the Quotient is 1.000069070; which being sought in the fourth Class gives 0, but being sought in the fifth Class gives 2, to be added to the Part of the Logarithm already found; and dividing the last Quotient by the Number last found in the Table, *viz.* 1.000046053, the Quotient is 1.000023015, which, being sought in the sixth Class, gives 9 to the Part of the Logarithm already found: And dividing the last Quotient by the new Divisor, *viz.* 1.000002072, the Quotient is 1.000000219, which being greater than 1.000000115 shews that the Logarithm already found, *viz.* 3.3010299 is less than the Truth by more than half an Unit; wherefore adding 1, you have *Briggs*'s Logarithm of 2000, *viz.* 3,3010300.

If any Logarithm be given, suppose 3,3010300, throw away the Characteristic, then overagainst these Figures 3... 0.. 1.. 0 .. 0, you have in their respective Classes 1,995262315..... 0..... 1,002305238 ..... 0 ..... 1,000069080 .... 0... 0 which multiplied continually into one another, the Product is 2,0000000019966, which, by reason the Characteristic is 3, becomes 2.00,00001 9966, &c. that is, 2000, the Natural Number desired. I shall not mention the Method by which this Table is framed, because you will easily see that from the Use of it.

It is obvious to the intelligent Reader, that these Classes of Numbers are no other than so many Scales of mean Proportionals: in the first Class, between 1 and 10; so that the last Number thereof, *viz.* 1,258925412 is the tenth Root of 10, and the rest in order ascending are the Powers thereof. So in the second Class, the last Number 1,023292992 is the hundredth Root of 10, and the rest in the same Manner are Powers thereof. So 1,002305238, in the third Class, is the tenth Root of the last of the second, and the rest its Powers, &c. Or, which is all one, each Number, in the

the preceding Class, is the tenth Power of the corresponding Number in the next following Class: Whence it is plain, that to construct these Tables requires no more than one Extraction of the fifth or sursolid Root for each Class, the rest of the Work being done by the common Rules of Arithmetic.

METHOD IV.

Their Construction, according to the common Rules, given by many Extractions of Roots, is tedious; the best Way yet known is this which follows.

To make a Table of Logarithms.

First, Put for the Logarithm of 1 a Cypher for the Index, and a competent Number of Cyphers for the Logarithm, according to the Number of Places you would have your Logarithms consist of; for 10 an Unit, with the same Number of Cyphers; for 100, 2, with as many Cyphers; for 1000, 3, with as many Cyphers, &c.

Secondly, Find the Difference between some two Logarithms above 1000, or rather above 10000, that differ by Unity; thus multiply the two Numbers together, and that Product you must multiply again by 43429448190325183896 * which last Product divided by the Arithmetical Mean between both Numbers, the Quotient is the Difference sought.

Suppose we would find the Difference between the Log. 10000, and 10001, the Product of these two Numbers is 1.00010000. which multiplied by 4343 produced 43434343; this divided by 10000,5, quotes 4343. Now if to the Logarithm of 10000, which is 4,0000000, you add the Difference before found, to wit, 434, the Sum 4.0000434 is the true Logarithm of 10001 to 7 Places.

Thirdly, Having thus found the Difference of any two Logarithms differing by Unity, and consequently some of the Logarithms by dividing the Difference found by the Arithmetical Mean, between any two Numbers differing by Unity, you shall have the Difference of the Logarithm of those two Numbers.

Thus to find the Difference betwixt the Logarithm of 274, and 275; divide 4343, the Difference of the Logarithm of 10000, and 10001 by 2745 the Quotient 15821, is the Difference sought.

Fourthly, Having by this Means found a few of the prime Logarithms, the rest are made by Addition and Subtraction, and hav-

* *Which is the Subtangent of the Curve expressing* Briggs's *Logarithms. See* Keil's *Tract.* Pag. 135, 140, &c.

ing

ing made the Canon upwards, above 1000 to 10000, by Consequence it is made for all inferior Numbers.

The prime Numbers to which Logarithms must be found, in the first Place are these, 2 . 3 . 7 . 11 . 13 . 17 . 19 . 23 . 29 . 31 . 37 . 41 . 43 . 47 . 53 . 59 . 61 . 67 . 71 . 73 . 79 . 89 . 97, &c. or the same Numbers with Cyphers.

But since it was very tedious and laborious, to find the Logarithms of the prime Numbers, and not easy to compute Logarithms by Interpolation, by first, second, and third, &c. Differences, therefore the great Men, Sir *Isaac Newton*, *Mercator*, *Gregory*, *Wallis*, and lastly, Dr. *Halley*, have published infinite converging Series, by which the Logarithms of Numbers to any Number of Places may be had more expeditiously and truer: Concerning which Series, Dr. *Halley* has written a learned Tract, in the *Philosophical Transactions*, wherein he has demonstrated those Series after a new Way, and shews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by Means of which may be found, easily and expeditiously, the Logarithms of large Numbers.

Let z be an odd Number, whose Logarithm is sought; then shall the Number $z-1$ and $z+1$ be even, and accordingly their Logarithms and the Difference of the Logarithms will be had, which let be called y: Therefore, also the Logarithm of a Number which is a Geometrical Mean between $z-1$ and $z+1$ will be given, viz. equal to the half Sum of the Logarithms. Now the Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25200z^9}$ &c. shall be equal to the Logarithm of the Ratio, which the Geometrical Mean between the Numbers $z-1$ and $z+1$, has to the Arithmetical Mean, viz. to the Number z.

If the Number exceeds 1000, the first Term of the Series $\frac{y}{4z}$ is sufficient for producing the Logarithm to 13 or 14 Places of Figures, and the second Term will give the Logarithm to 20 Places of Figures. But if z be greater than 10000, the first Term will exhibit the Logarithm to 18 Places of Figures; and so this Series is of great Use in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example, It is required to find the Logarithm of 20001. The Logarithm of 20000 is the same as the Logarithm of 2, with the Index 4 prefixed to it; and the Difference of the Logarithms of 20000 and 20002, is the same as the Difference of the Logarithms of the Numbers 10000 and 10001, viz. 0.00004342 727687. And if this Difference be divided by $4z$, or 80004, the Quo-

Quotient $\frac{y}{4z}$ shall be ———————— 0. 00000 0000542813

And if the Logarithm or the Geometrical Mean be added to the Quotient, the Sum will be the Logarithm of 20001. Wherefore it is manifest, that

4. 30105 1709302416

4. 30105 1709845230

to have the Logarithm to 14 Places of Figures, there is no Necessity of continuing out the Quotient beyond 6 Places of Figures. But if you have a Mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlacq*'s Tables, the two first Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000 are to be found by this Way, the Labour of doing them will mostly consist in setting down the Numbers. *Note*, This Series is easily deduced from that found out by Dr. *Halley*; and those who have a Mind to be informed more in this Matter, let them consult his abovenamed Treatise.

Mr. *WARD*'s *Easy* Method of making the Canon of Sines, Tangents, &c.

FIRST, let me premise two Things, that the Periphery of a Circle, whose Radius is Unity or 1, is 6.283185, &c. and that the natural Sine of one Minute doth so insensibly differ from the Length of the Arch of one Minute, that it may be taken for the same.

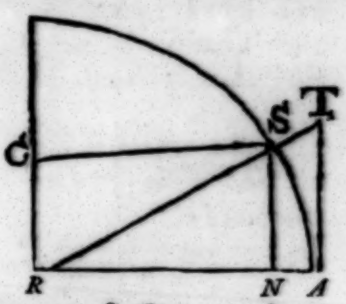
Consequently, { *As the Periphery in Minutes : is to the Periphery in equal Parts of the Radius : : so is one Minute : to the Parts agreeing to that Minute.*

That is, 21600′ : 6,283183 : : 1′ : 0,000290888 = the Natural Sine of one Minute; which agrees with the largest Table of Sines I ever saw.

Having thus got the Sine of one Minute, its Co-sine may bo thus found:

Suppose

Suppose $RA = RS$ the Radius of any Circle, $SN =$ the Sine of the Arch SA. Then $RN = CS$ is the Co-sine of that Arch. But $\square RS - \square SN = \square RN$, consequently, $\sqrt{\square RS - \square SN} = RN$.



That is, from the Square of the Radius, subtract the Square of the Sine of 1′, the square Root of the Remainder will be the Co-Sine of 1′, per Chap. 9. Prop. 1. In Numbers, the Sine of 1′ is 000290885, its Square is 0,0000000084612; and 1—0,0000000084612 = 0,9999999915388, the Square Root thereof is ,99999995 = the Co-Sine required.

The Sine and Co-Sine of one Minute being thus obtained, all the rest of the Sines in the Quadrant may be gradually calculated by Mr. *Michael Dary*'s Sinical Proportions; which I shall here insert, to the same Effect as they are in his Miscellanies; and then explain and demonstrate the Truth of those Proportions.

If a Rank of Arches be equi-different;

Then { " As the Sine of any Arch in that Rank : is to the Sum of the Sines of any two Arches equally remote from it on each Side : : so is the Sine of any other Arch in the said Rank : to the Sum of the Sines of two Arches next it on each Side, having the same common Distance.

Immediately after these Proportions, he lays down the following Equations:

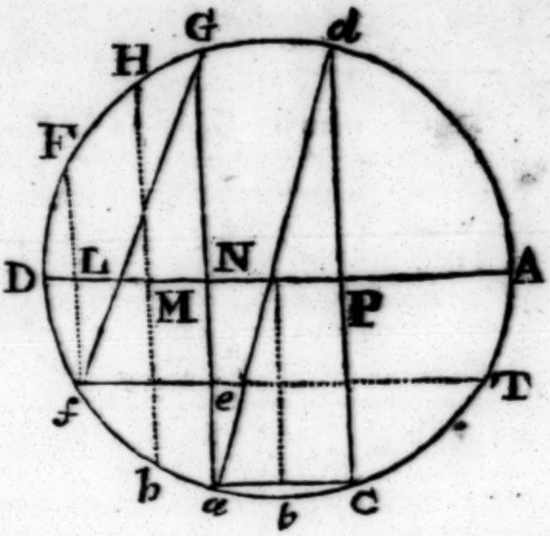
Three Arches equi-different being proposed; if (saith he) you put $Z =$ the Sine of the great Extreme, $y =$ the Sine of the lesser Extreme; $M =$ the Sine of the Mean; $m =$ the Co-sine thereof; D the Sine of the common Difference; $d =$ the Co-sine thereof; and $R =$ the Radius.

1. The $Z + y = \frac{2Md}{R}$. 2. Then $Z - y = \frac{2mD}{R}$.

3. Then $Zy = MM - DD$. 4. Then $\frac{Z}{R} = \frac{Md + mD}{Md - Md}$

From the foregoing it is evident (saith he) that if two Thirds, viz. either the former or latter 60 Degrees, or the former 30 Deg. and the latter 30 Degr. of the Quadrant be completed with Sines; the remaining Part of the Quadrant may be completed by Addition, or Subtraction only.

Thus far is from the ingenious Mr. *Dary*, concerning these excellent Proportions; the Truth whereof I shall thus demonstrate.



In the annexed Circle $DA = da$ are Diameters, $fb = ba = ab = bc$, are equal Arches.

Draw fT parallel to DA; then will $Ne = Lf$. And the $\triangle\ dac$, like the $\triangle\ Gfe$, being both right-angled at c and e, and $< d = < G$ because subtended by the equal Arches $ac = fa$.

Therefore $da : dc :: Gf : Ge$.

Consequently $\frac{1}{2}\,da : dc :: \frac{1}{2}\,Gf : Ge$. But $Hb = Gf$, whence $HM = \frac{1}{2}\,Gf$, and $\frac{1}{2}\,da =$ the Radius, $dp = \frac{1}{2}\,dc$. Therefore it will be, Radius : $2\,dp = :: HM = \frac{1}{2}\,Gf : GN + Ne = GN + Lf$. That is, as the Radius : is to twice the Sine dp :: so is the Sine HM : to the Sum of the two Sines GN and $FL = fL$.

Q. E. D.

I shall now explain these proportions, and shew how they may be applied in Practice: Having the Sine of one Minute, and its Co-sine as before; let the Radius be made the mean or middle Term between those two Extremes; then the Proportions will run

Thus { " As the Radius : is to the double Co-sine of one Minute :: so is the Sine of one Minute : to the Sine of two Minutes, and of 00″: and so is the Sine of 2′ :: to the Sum of the Sines of 3′ and 1′ :: and so is the Sine of 3′ : to the Sum of the Sine of 4′ and 2′."

And so on in a successive Order, from Minute to Minute.

And then, if from the Sum of the Sines of 3′ and 1′ be taken the Sine of 1′, the Remainder will be the Sine of 3′: And the like, if, from the Sum of the Sines of 4′ and 2′, be taken the Sine of 2′, there will remain the Sine of 4′, &c.

Proceeding on by this Method, all the natural Sines in the Quadrant may be easily calculated by Addition, and Subtraction only. For the Radius, or first Term in the Proportion, being 1,0000000 or Unity, Division is wholly avoided: And because the second Term in the Proportion varies not, if a Tariffa, or small Table be made thereof, to all the nine Digits, then Multiplication

is

is also avoided. For, by the Help of that Tariffa, the whole Work may be performed by Addition and Subtraction, until all the Sines are gradually made.

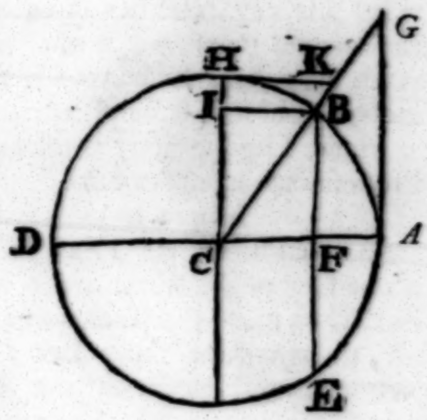
Thus you have an easy Way of making the Canon of Sines; which being once done, the Tangents and Secants may be found by the following

Proportions { *As the Co-sine of any Arch : is to the Sine of that Arch : : so is the Radius : to the Tangent of the same Arch.*

That is, by *the first Scheme of this Problem*, $RN:SN::R:TA$. And $RN:RS::RA:RT =$ the Secant of that Arch.

Plane Trigonometry.

Definitions.



1. A Circle is supposed to be divided into 360 equal Parts, called Degrees; and each Degree into 60 equal Parts, called Minutes; and each Minute into 60 equal Parts, called Seconds, &c. Any Portion of whose Circumference is called an Arch, and is measured by the Number of Degrees it contains.

2. A Chord or Subtense is a straight Line, connecting the Extremities of an Arch; as *BE* is the Chord of the Arches *BAE*, *BDE*.

3. A Sine (or Right-sine) is a straight Line drawn from one End of an Arch perpendicular to that Diameter passing through the other End; or it is half the Chord of twice the Arch; so *BF* is the Sine of the Arches *BA*, *BD*. And here it is evident, that the Sine of 90 Degrees (which is equal to the Radius or Semi-Diameter of the Circle) is the greatest of all Sines, the Sine of an Arch greater than a Quadrant being less than the Radius.

4. The Difference of an Arch from a Quadrant, whether it be greater or less, is called its Complement; so *HB* is the Complement of the Arches *BA*, *BD*; *BI* is the Sine of that Complement, and therefore it is called the Co-sine, or Sine-Complement of the Arches *BA*, *BD*.

5. The Secant of an Arch is a straight Line drawn from the Center thro' one End of the Arch till it meet with the Tangent, which

which is a straight Line touching the Circle at the Extremity o that Diameter which cuts the other End of the Arch; so *CG* is the Secant, and *AG* the Tangent of the Arches *BA, BD*: And *CK* is the Co-secant, aud *HK* the Co-tangent of the said Arches.

6. A versed Sine is the Segment of the Diameter intercepted between the Arch and its Sine: Thus *FA* is the versed Sine of the Arch *BA*, and *FD* of the Arch *BD*.

7. Whatever Number of Degrees an Arch wants of a Semi-Circle is called its Supplement.

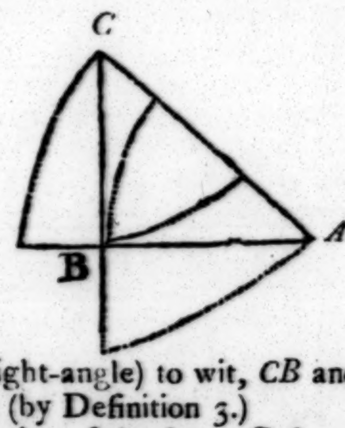
8. That Part of the Radius which is betwixt the Center and Sine is equal to the Co-sine; thus *CF* is $=$ *IB*.

2. If an Arch be greater or less than a Quadrant, the Sum or Difference of the Radius or Co-sine is equal to the versed Sine.

In a Triangle are six Parts, *viz.* three Sides and three Angles: Any three of which being given, except the three Angles of a Plane Triangle, the other three may be found either Mechanically, by the Help of a Scale of equal Parts and Line of Chords, or by an Arithmetical Calculation, if, supposing the Radius divided into any Number of equal Parts, we know how many of those equal Parts are in the Sine, Tangent, or Secant of any Arch proposed: The Art of inferring which is called Trigonometry, and it is either Plane or Spherical.

Plane Trigonometry is solved by the Help of four fundamental Propositions called Axioms.

Axiom I.



In a Right-angled Triangle *ABC*, if one Leg of the Right-angle, as *AB* or *CB*, be made the Radius of a Circle, then shall the other Leg *CB* or *AB* be the Tangent of the Angle opposite to it, and the Hypothenuse *AC* (or Side opposite to the Right-angle) its Secant (by Definition 5.)

But if the Hypothenuse *AC* be made the Radius of a Circle, then will the Legs (or Sides including the Right-angle) to wit, *CB* and *AB* be the Sines of the Angles opposite (by Definition 3.)

Upon this Axiom depends the Solution of the seven Cases of Right-angled Plane Triangles.

Note, That the three Angles of a Plane Triangle make two Right-angles, or 180 Degrees, by 32. 1 Eucl.

For the more easy making the Proportions for the Solution of Right-angled Triangles, observe, that as different Sides are made Radius, so the other Sides require different Names, which Names are either Sines, Tangents, or Secants, and are to be taken out of your Table.

To

To find a Side, any Side may be made Radius: Then ſay, as the Name of the Side given is to the Name of the Side required; ſo is the Side given to the Side required.

But to find an Angle, one of the given Sides muſt be made Radius; then, as the Side made Radius, is to the other Side; ſo is the Name of the firſt Side (which is Radius) to the Name of the ſecond Side; which fourth Proportional muſt be found among the Sines or Tangents, &c. to be determined by the Side made Radius, and againſt it is the Angle required.

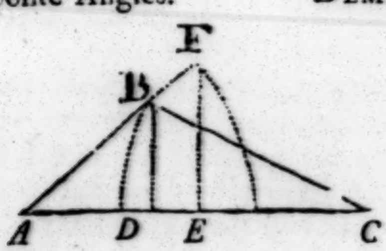
The Proportion for the Solution of ſeven Caſes of Plane Right-angled Triangles. [*See the next foregoing* Fig.]

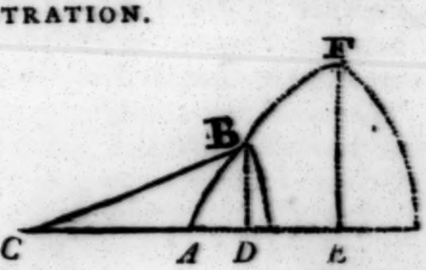
Given	*Reqd.*	*Proportions.*	*Rad.*	*Caſe.*
AB A and C	BC	Coſi. A : Si. A : : AB : BC. R : Tan. A : : AB : BC. Co-t. A : R : : AB : BC.	AC AB BC	1
AB A and C	AC	Coſi. A : R : : AB : AC. R : Sec. A : : AB : AC. Tan. A : Coſe. A : : AB : AC.	AC AB BC	2
AB BC	A and C	AB : BC : : R : Tan. A. Complement is C. BC : AB : : R : Tan. C. Complement is A.	AB BC	3
AB BC	AC	AB : BC : : R : Tan. A; then Coſi. A : R : : AB : AC, or $\surd:\overline{\square AB+\square BC}:=AC$ (*per* 47 1. Eucl.	AB AC	4
AB AC	A and C	AC : BC : : R : Coſi. A. AB : AC : : R : Secant A.	AC AB	5
AB AC	BC	AC : AB : : R : Coſi. A; then R : Tan. A : : AB : BC, or $\surd:\square AC-\square AB:=BC$.	AC AB	6
AC A and C	AB	C : Coſi. A : : AC : AB. Sec. A : R : : AC : AB. Coſ. A : Coſ. A : : AC : AB.	AC AB BC	7

Axiom II.

In any Triangle the Sides are proportional to the Sines of the oppoſite Angles.

Demonstration.





Produce

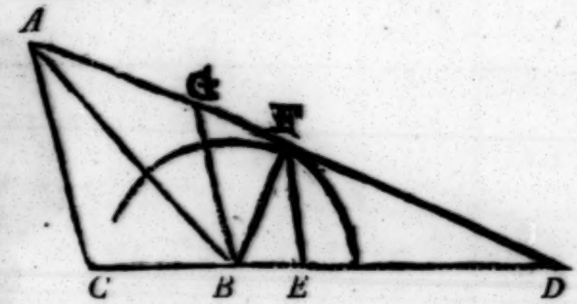
Produce the leffer Side of the Triangle *ABC*, to wit *AB* to *F*, making $AF=BC$: Let fall the Perpendiculars *BD*, *FE*, upon the Side *CA* produced if Need be; then will *FE* be the Sine of the Angle *A*, and *BD* the Sine of the Angle *C*, to the Radius $BC=AF$.

Now the Triangles *ABD* and *AFE*, having the $< A$ common to them both, and the $< D = < E =$ to a Right-angle, are fimilar; wherefore (by 4. 6 *Eucl. Elem.*) $AF (BC) : AB :: FE : BD$; *viz.* :: *Si. A* : *Si. C*. Q. E. D. Otherwife thus; by Ax. I. $AB : R :: BD : Si.\ A$, and $BC : R :: BD : Si.\ C$; therefore $AB \times Si.\ A (= R \times BD) = BC \times Si.\ C$; wherefore $AB : BC :: Si.\ C : Si.\ A$. Q. E. D.

Axiom III.

The Sum of the Legs of any Angle of a Plane Triangle is to their Difference, as the Tangent of half the Sum of the Angles oppofite to thofe Legs is to the Tangent of half their Difference.

Demonstration.

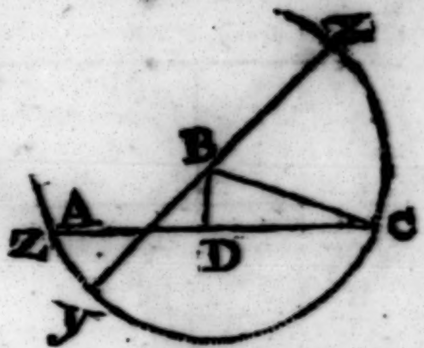


In the Triangle *ABC* produce *CB*, the leffer Leg of the Angle *B*, till *BD* becomes $= BA$, the greater Leg, and then bifect *CD* in *E*; join *AD* and bifect it alfo in *F*; draw *BF*, which (by 8. 1 *Eucl. El.*) will be perpen. to *AD*; and draw *EF*, which (by 2. 6 *Eucl. Elem.*) will be parallel to *AC*. Then will the Angle $ABF = FBD = \frac{1}{2} ABD$, which external Angle *ABD* is (by 32. 1 *Eucl. Elem.*) $= BAC + C$, that is to the Sum of the oppofite Angles required.

Draw then *BG* parallel to *AC*, fo will the Angle *GBA* be (by 29. 1 *Eucl. Elem.*) equal to its alternate one *BAC*; and if from half the Sum of the oppofite Angles you take the leffer Angle, i. e. If from $< ABF$ you take the $< GBA$, there will remain $< GBF =$ half the Difference of the oppofite Angles: And fo alfo, if from *CE* half the Sum of the Legs, you take *CB* the leffer Leg, there will remain *BE* equal to half the Difference of the Legs. And then fince the $\triangle ABF$ is Right-angled, if *BF* be made Radius, *AF* will be the Tangent of $< ABF$ (*i. e.* the Tangent of half the Sum of the oppofite Angles;) and in the little $\triangle GBF$, *FG* will be the Tangent of the $< GBF$ (*i. e.* the Tangent of half the Difference of the oppofite Angles:) But the Segments of the Legs of any Triangle cut by Lines parallel to the Bafe, being (by Schol. to 2. 6 *Eucl. El.*) proportional; $EC : EB :: FA : FG$; that is in Words, half the Sum of the Legs is to half their Difference, as the

the Tangent of half the Sum of the opposite Angles is to the Tangent of half their Difference: But Wholes are as their Halves: whereof the Sum of the Legs is to their Difference, as the Tangent of half the Sum of the Angles opposite is to the Tangent of half their Difference. Q. E. D.

Axiom IV.



The Base or greatest Side of any Plane Triangle is to the Sum of the Legs, as the Difference of the Legs is to the Difference of the Segments of the Base made by a Perpendicular let fall from the Angle opposite to the Base.

Demonstration.

From the $< B$, on the Base AC, of the $\triangle ABC$, let fall the Perpendicular BD; on B. as a Center, with the greater Leg AC, as a Radius, describe the Circle $BxCyZ$; and produce AB to x and y, and CA to Z. Then by the 35. 3 *Eucl. Elem.* $Ay \times Ax$ is $= AC \times AZ$; viz.: $BC - BA : \times : BC + BA : = AC \times : DC - DA$: therefore $AC : BC + BA :: BC - BA : DC - DA$. Q. E. D. Otherwise, let the Difference of the Squares of the Sides BC and AB be taken and divided by the Base AC, the Quotient shall be the Difference of the Segments of the Base aforesaid: Or, Square all the 3 Sides, and deduct the Square of one of the less Sides out of the Sum of the other two Squares, divide half the Remainder by the longest Side, the Quotient is the alternate Segment of the Base.

The Proportions for the Solution of the six Cases of Plane oblique Triangles. [*See the last* Fig.]

Given.	*Reqd.*	*Proportions.*	*Ax.*	*Case.*
A B B C and C	A	AB : BC : : Si. C : Si. A.	2	1

N. B. 1st. If the given Angle be obtuse, the other 2 Angles then are each acute.

2dly, If the Side opposite to the given Angle is longer than the Side opposite to the Angle sought, then is the Angle sought acute; but if shorter, then is the said Angle doubtful, and may be either acute or obtuse, because both the Sine and its Complement to two right Angles are the same: Wherefore to be certain, of what Quality the Angle opposite to the greatest Side is: Take the Sum and Difference of the greatest Side and Midde (or least) add their Logarithms, if the half of them be equal to the Logarithm of the third Side, the Angle opposite to the greatest Side is a right Angle; but if

if the Logarithm of the third Side be greater than the half, it is Acute; if lefs, it is obtufe: Or, without Logarithms, multiply the faid Sum by the Difference abovefaid, and extract the Square Root. which if {Equal to / Greater than / Lefs than} the third Side, then is the greateft Angle {Right / Obtufe / Acute}

Given.	*Reqd.*	*Proportions.*	*Ax.*	*Cafe.*
AB BC and C	AC	AB : BC : : Si. C : Si. A. Hence, by Subtraction, the $<$ B will be known. Si. A : Si. B : : BC : AC.	2	2
A, C & BC	A	Si. A : Si. C : : BC : AB.	2	3
B AB BC	A and C	BC + AB : BC — AB : : Tan. $\frac{1}{2}$ Sum of the $<$s oppofite : Tan. $\frac{1}{2}$ Diff. of the $<$s oppofite. Then $\frac{1}{2}$ Sum + $\frac{1}{2}$ Differ. = greater $<$ A; and $\frac{1}{2}$ Sum — $\frac{1}{2}$. Difference = leffer $<$ C.	3	4
B AB BC	AC	Firft, find the Angles by the laft; then Si. C : Si. B : : AB : AC	3 2	5
AB BC AC	A B C	AC : BC + BA : : BC — BA : DC — DA : Then $\frac{1}{2}$ AC + $\frac{1}{2}$ DC — $\frac{1}{2}$ DA = DC. And $\frac{1}{2}$ AC — : $\frac{1}{2}$ DC — $\frac{1}{2}$ DA : = DA.	4	6
		Then AB : AD : : R : Cofi. A.	1	
		And CB : DC : : R : Cofi. C.	1	
		And 180° — $<$ A — $<$ C = $<$ B.		

Or more readily at one Operation.

From half the Sum of the Sides fubduct each particular Side, and let the Sum of the Logarithm of the half Sum and Difference of the Side fubtending the enquired Angle be fubducted from the Sum of the Log. of the other Difference and the double Radius, half the Remainder fhall be the Log. of the Tangent of half the enquired Angle.

Agreeable to this Axiom in *Gellibrand*'s Trig. Britannica, p. 46.

"As the Rectangle of half the Sum of the Sines and the Difference between that half Sum and the Side oppofite to the Angle required, is to the Rectangle of the other two Remainders; fo is the Square of Radius to the Square of the Tangent of half the Angle fought."

Ex Angulis latera, vel ex lateribus Angulos & mixtim in Triangulis tam planis quam Sphæricis affequi, Summa Gloria Mathematici eft: Sic enim Cælum & Terras & Maria felici & admirando calculo menfurat. Fran. Vieta.

F I N I S.

THE

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